0.1 The Eigenvalue Problem

$$\left.\begin{array}{l} \dfrac{d^2\Psi(x^*)}{dx^{*\,2}} - l^2\Psi(x^*) = 0, \quad \Psi(0) = 0\ , \Sigma(1) = 0, \\ \\ \dfrac{d^2\Sigma(x^*)}{dx^{*\,2}} - l^2\Sigma(x^*) = 0,\ \Psi(S_0^*) = \Sigma(S_0^*) + \Phi\, N_0, , \end{array}\right\} \tag{1}$$

The solutions of (1) for the pressure eigenfunctions subject to the pressure continiuty condition (1) are

$$\left.\begin{array}{l} \Psi(x^*) = 2\,C_3\,\sinh(\,l\,x^*), \\ \\ \Sigma(x^*) = 2\,C_4\,\dfrac{\sinh(\,l\,(x^* - 1))}{\cosh(\,l) - \sinh(\,l)}, \end{array}\right\} \tag{2}$$

where the constant of integration are

$$C_3 = \frac{1}{2}\frac{\Phi}{l}\left[\frac{\left\{\varphi\,\sigma\,(1 - R_1) + \coth(l\,(S_0^* - 1))\,l\,N_0\right\}}{\sinh(l\,S_0^*)\left\{R\,\coth(l\,(S_0^* - 1)) - \coth(l\,S_0^*)\right\}}\right],$$

$$C_4 = \frac{1}{2}\frac{\Phi}{l}\left[\frac{\left\{\varphi\,\sigma\,(1 - R_1) + \coth(l\,S_0^*)\,l\,N_0\right\}\left\{\cosh(l) - \sinh(l)\right\}}{\sinh(l\,(S_0^* - 1))\left\{R\,\coth(l\,(S_0^* - 1)) - \coth(l\,S_0^*)\right\}}\right].$$

where

$$N_0 = F_1(S_0^*)\,\frac{C}{k}\,\frac{Pec_i}{R}\left\{(R - 1) + \frac{k\,R\,R_3\,(1 - R_1)}{C\,Pec_i}\right\}.$$

$$F(S_0^*) = \frac{\dot{m}_{liq,vap}}{\dot{m}_i} = F_1(S_0^*)\,[1 - R\,G_1\,S_0^*]\,, \quad F_1(S_0^*) = \frac{1}{R\,S_0^* + 1 - S_0^*},$$

$$G_1 = \frac{R_3\,k(1 - R_1)}{C\,Pec_i}. \tag{3}$$

The eigenvalue problem for the temperature distributions in the both phases,

$$\left.\begin{aligned}
&\left(\frac{d^2}{dx^{*\,2}} - \omega_0\,\frac{d}{dx^*} - E_1\,\sigma - l^2\right)\phi_{liq} + \frac{d\Theta^0_{liq}}{dx^*}\,\frac{d\Psi}{dx^*} = 0,\\
&\phi_{liq}(0) = 0, \quad \phi_{liq}(S^*_0) = -\Phi\,\left.\frac{d\Theta^0_{liq}}{dx^*}\right|_{x^*=S^*_0},\\
&\left(\frac{d^2}{dx^{*\,2}} - \Omega_0\,\frac{d}{dx^*} - \frac{E_2\,k\,R_1}{C}\,\sigma - l^2\right)\phi_{vap} + \frac{R\,k}{C}\,\frac{d\Theta^0_{vap}}{dx^*}\,\frac{d\Sigma}{dx^*} = 0,\\
&\phi_{vap}(1) = 0, \quad \phi_{vap}(S^*_0) = -\Phi\,\left.\frac{d\Theta^0_{vap}}{dx^*}\right|_{x^*=S^*_0},
\end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned}
&\frac{d\Theta^0_{liq}}{dx^*} = N_1\,\exp\left(\omega_0\,x^*\right),\\
&\frac{d\Theta^0_{vap}}{dx^*} = N_2\,\exp\left(\Omega_0\,(x^*-1)\right),
\end{aligned}\right\} \tag{5}$$

with leads to

$$\left.\frac{d\Theta^0_{liq}}{dx^*}\right|_{x^*=S^*_0} = N_3, \quad \text{and} \quad \left.\frac{d\Theta^0_{vap}}{dx^*}\right|_{x^*=S^*_0} = N_4, \tag{6}$$

where

$$\left.\begin{aligned}
&N_1 = \frac{\omega_0}{\exp(\omega_0\,S^*_0)-1},\ N_2 = \frac{\Omega_0}{\exp(\Omega_0\,(S^*_0-1))-1},\\
&N_3 = N_1\,\exp(\omega_0\,S^*_0), \quad N_4 = N_2\,\exp\left(\Omega_0\,(S^*_0-1)\right).
\end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned}
&\omega_0 = \frac{1}{k}\,Pec_i\,F(S^*_0)\,C,\\
&\Omega_0 = Pec_i\,F(S^*_0),
\end{aligned}\right\} \tag{8}$$

The solution of (4) for the eignfunction of the liquid temperature distribution is

$$\begin{aligned}
\phi_{liq}(x^*) =& \frac{C_3\,l\,N_1}{f_1\,f_4\,f_5\,f_6}\left\{f_1\,f_4\,f_6\,\exp(x^*(\omega_0-1)) + \left\{2\,f_1\,f_4\,E_1\,\sigma\,\cosh\left(\frac{\gamma_3\,x^*}{2}\right)\right.\right.\\
&\left.-\left[2\,f_1\,f_2\,E_1\,\sigma + f_6\,f_7 - f_3\,f_5\right]\sinh\left(\frac{\gamma_3\,x^*}{2}\right)\right\}\exp\left(\frac{x^*\,\omega_0}{2}\right)
\end{aligned}$$

$$-f_1\, f_4\, f_5\, \exp(x^*(\omega_0+1))\Big\} - \frac{\Phi\, N_3}{f_1\, f_4} \exp\left(\frac{x^*\,\omega_0}{2}\right) \sinh\left(\frac{\gamma_3\, x^*}{2}\right), \tag{9}$$

where

$$\gamma_3 = \sqrt{\omega_0^2 + 4\, E_1\, \sigma + 4\, l^2}, \quad f_1 = \exp\left(\frac{\omega_0\, S_0^*}{2}\right), \quad f_2 = \cosh\left(\frac{\gamma_3\, S_0^*}{2}\right),$$

$$f_3 = \exp\left((\omega_0 + l)\, S_0^*\right), \quad f_4 = \sinh\left(\frac{\gamma_3\, S_0^*}{2}\right), \quad f_5 = (\omega_0\, l\, + E_1\, \sigma\,),$$

$$f_6 = (\omega_0\, l\, - E_1\, \sigma\,), \quad f_7 = \exp(S_0^*\,(\omega_0 - l)).$$

The solution of (4) for the vapour temperature distribution is

$$\begin{aligned}\phi_{vap}(x^*) =& \frac{C_4\, N_2}{f_{19}} \Big\{ f_{20} \exp\left((\Omega_0 - l)x^* - \Omega_0 + 2\, l\right) + f_{21} \exp\left(\frac{\Omega_0\, x^*}{2}\right) \sinh\left(\frac{\gamma_4\, x^*}{2\, C}\right) \\ & - f_{22} \Big\{ f_{23}\, \exp\left(\frac{\Omega_0\, x^*}{2}\right) \cosh\left(\frac{\gamma_4\, x^*}{2\, C}\right) + f_{13} \exp((\Omega_0 + l)\, x^* - \Omega_0) \Big\}\Big\} \\ & - \Phi\, N_2\, f_{24}\, \exp\left(\frac{x^*\, \Omega_0}{2}\right) \sinh\left(\frac{\gamma_4\, x^*}{2\, C}\right), \end{aligned} \tag{10}$$

where

$$\gamma_4 = \sqrt{((\Omega_0^2 + 4\, l^2)\, C^2 + 4\, E_1\, \sigma\, C\, k\, R_1}, \quad f_8 = (C\, \Omega_0\, l\, - k\, R_1\, E_1\, \sigma\,),$$

$$f_9 = \exp\left(\frac{\Omega_0\, S_0^*}{2}\right), \quad f_{10} = \sinh\left(\frac{\gamma_4\, S_0^*}{2\, C}\right), \quad f_{11} = \exp\left(-\Omega_0\right),$$

$$f_{12} = \exp\left(-\Omega_0 + 2\, l\right), \quad f_{13} = (C\, \Omega_0\, l\, + k\, R_1\, E_1\, \sigma\,), \quad f_{14} = \cosh\left(\frac{\gamma_4\, S_0^*}{2\, C}\right),$$

$$f_{15} = \exp\left(\Omega_0 (S_0^* - l)\right), \quad f_{16} = \exp\left((\Omega_0 + l)\, S_0^* - \Omega_0\right),$$

$$f_{17} = \exp\left(\Omega_0\, (S_0^* - 1) - l\, (S_0^* - 2)\,\right), \quad f_{18} = E_1^2\, \sigma^2 + C^2\, \Omega_0^2\, l^2 - k^2\, R_1^2,$$

$$f_{19} = f_9\, f_{10}\, f_{18}, \quad f_{20} = f_{22}\, f_8, \quad f_{22} = R\, k\, l\, f_9\, f_{10}, \quad f_{24} = \frac{f_8\, f_{13}\, f_{15}}{f_{19}},$$

$$f_{21} = R\,k\,l \left\{ f_9\, f_{12}\, f_{14} (C\,\Omega_0\, l - \; R_1\, E_1\, \sigma\, k) - f_8\, f_{17} - f_9\, f_{11}\, f_{13}\, f_{14} + f_{13}\, f_{16} \right\},$$

$$f_{23} = \left\{ f_{12}\, (C\,\Omega_0\, l - \; R_1\, E_1\, \sigma\, k) - f_{11}\, f_{13} \right\}.$$

0.2 Dispersion Analysis

An equation which represent the relationship between the decay rate σ and the wave number l is known as dispersion equation and can be obtained by using the energy jump across the liquid-vapour interface.

$$\varphi\, \sigma\, H_{liq}\, \Phi = \left\{ \Phi\, \frac{d^2\Theta^0_{liq}}{dx^{*\,2}} + \frac{d\phi_{liq}}{dx^*} \right\} + \frac{\Theta_0}{k} \left\{ \Phi\, \frac{d^2\Theta^0_{vap}}{dx^{*\,2}} + \frac{d\phi_{vap}}{dx^*} \right\} - H_{liq}\, \frac{d\Psi_1}{dx^*}. \tag{11}$$

From the basic state we know that

$$\left. \begin{aligned} \left.\frac{d^2\Theta^0_{liq}}{dx^{*\,2}}\right|_{x^*=S^*_0} &= N_3\, \omega_0, \\ \left.\frac{d^2\Theta^0_{vap}}{dx^{*\,2}}\right|_{x^*=S^*_0} &= N_4\, \Omega_0. \end{aligned} \right\} \tag{12}$$

Substituting (12) into (11) yields

$$\varphi\, \sigma\, H_{liq}\, \Phi = \left\{ \frac{d\phi_{liq}}{dx^*} + \Phi\, N_3\, \omega_0 \right\} + \frac{\Theta_0}{k} \left\{ \frac{d\phi_{vap}}{dx^*} + \Phi\, N_4\, \Omega_0 \right\} - H_{liq}\, \frac{d\Psi_1}{dx^*}. \tag{13}$$

The complete solution for $\sigma(l)$ has been obtained by substituting (2), (9) and (10) into (13) which is solved numerically using Maple.

I want to solve (13) for $\sigma(l)$ considering two cases

1. taking limit of (13) when $\sigma \to 0$ and $l \to 0$ to find R_3. using this R_3 and will plot $\sigma(l)$ to check the stability

2. taking limit of (13) when $\sigma \to 0$ and $l \to \infty$ to find R_3. using this R_3 and will plot $\sigma(l)$ to check the stability

If you have time then plz help me to find a way to get the solution in more readable form.

Note

The value of S_0^* show be obtained first from the following equation

$$\frac{\Theta_0}{C} = \{\exp\left[Pec_i\, F(S_0^*)\,(1 - S_0^*)\right] - 1\} \left\{ H_{liq} + \frac{1}{1 - \exp\left[-\dfrac{C\, F(S_0^*)\, Pec_i\, S_0^*}{k}\right]} \right\} \tag{14}$$

The values of the parameters are parvals:= $\varphi = 0.38,\quad R_1 = 0.0009,\quad R_2 = 8.75,\quad E_1 = 1,\quad E_2 = 1,$

$C = 1.96,\quad k = 4,\quad H_{liq} = 7,\quad Pec_i = -120,\quad \Phi = 1,\quad \Theta_0 = 1.9, R = 20$