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An analytical solution for nonlinear dynamics of a viscoelastic beam-heavy mass system †

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Abstract

The aim of this study is to develop an approximate analytic solution for nonlinear dynamic response of a simply-supported Kelvin-Voigt viscoelastic beam with an attached heavy intra-span mass. A geometric nonlinearity due to midplane stretching is considered and Newton's second law of motion along with Kelvin-Voigt rheological model, which is a two-parameter energy dissipation model, are employed to derive the nonlinear equations of motion. The method of multiple timescales is applied directly to the governing equations of motion, and nonlinear natural frequencies and vibration responses of the system are obtained analytically. Regarding the resonance case, the limit-cycle of the response is formulated analytically. A parametric study is conducted in order to highlight the influences of the system parameters. The main objective is to examine how the vibration response of a plain (i.e. without additional adornment) beam is modified by the presence of a heavy mass, attached somewhere along the beam length.

Keywords: Nonlinear dynamics; Kelvin-Voigt viscoelastic material; Method of multiple timescale

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1. Introduction

The demand for structural members is continuously increasing, largely due to their growing application in many technological devices and machine components. Many examples of which can be found in the frame of aerospace vehicles, bridges, and automobiles. In some of these applications, a commonly occurring situation is that a uniform beam is subjected to several adornments, e.g. heavy masses or springs, at several locations along its length. Due to the widespread application, many investigators have studied the vibration responses of this class of systems (see Ref. [1]).

As reviewed by Nayfeh and Mook [1], nonlinear vibrations of plain beams (i.e. without additional adornments) have received considerable attention for many years, for example, by Woinowski-Krieger [2], Burgreen [3], Eisley [4], Srinivasan [5]. Further work on the subject involved, for instance, the inclusion of rotary inertia and transverse shear by Wrenn and Meyers [6], buckled beam by Tseng and Dugundji [7], and nonlinearity due to the nonlinear curvature-displacement relationship by Hu and Kirmser [8].

The first study on the dynamics of the system in the presence of additional adornments along the beam length was conducted by Dowell [9], who found that the influence of nonlinear spring–mass system on the vibrations of a simply-supported beam is important. Further studies on this topic have been conducted by Birman [10] and Szemplinska-Stupnicka [11].

The work of Dowell [9] was extended by Pakdemirili and Nayfeh [12] who considered the midplane stretching effect. This work was extended recently by Pakdemirili and Boyaci [13] and Ozkaya and Pakdemirili [14] for a system with both ends clamped, using a perturbation technique.

In the modeling of engineering structures, an important issue is how to model the material of the structures. All the references cited above considered only elastic models; however, in some special applications, such as artificial muscles [15] and some special vibration absorbers, the viscosity of the structure is the dominant characteristic of the system [15-18]. Therefore, in the present paper, the nonlinear vibrations of a Kelvin-Voigt viscoelastic beam with a heavy[1] intra-span mass is investigated. The equations of motion along with the corresponding boundary and time-dependent compatibility (internal

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[1] By saying *heavy*, we imply that in the equations of motion the order of magnitude of terms including point-mass effects is as same as that of dominant linear terms; this makes the solution procedure different from a straightforward application of the method of multiple timescale to a beam model.

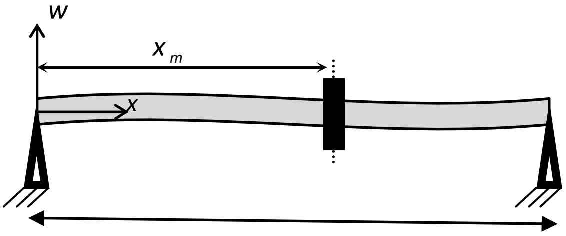

Fig. 1. A Kelvin-Voigt viscoelastic beam with a intra-span mass.

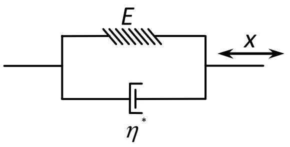

Fig. 2. The Kelvin-Voigt viscoelastic material model.

boundary) conditions are derived via Newton's second law of motion and by employing the Kelvin-Voigt rheological, energy dissipation model. Closed form solutions for nonlinear natural frequencies, nonlinear vibration responses, and frequency-response relations are obtained via a perturbation technique. Numerical examples, illustrating the effects of system parameters on the above-mentioned vibration characteristics of the system is given in the last section of this study.

2. Mathematical model and the equation of motion

The weakly nonlinear equations of motion for a Kelvin-Voigt viscoelastic beam, with a heavy, intra-span mass are derived in this section using Newton's second law of motion[2].

As shown in Fig. 1, a Kelvin-Voigt viscoelastic beam of length L, constant density ρ, cross-sectional area A, and Young's modulus E is considered. There is a heavy point mass, M, attached to the beam at x_m from the left-hand end. The beam is considered as a two-part system; namely the span before the mass ($0 < x < x_m$) and the span after ($x_m < x < L$). At the boundary between these two spans ($x = x_m$), the point-mass is attached. Moreover, w_1 and w_2 represent respectively the displacements for these two spans.

Considering only the transverse displacement, as well as using Kelvin-Voigt rheological model of material (Fig. 2), the constitutive relation for each span is given by

$$(\sigma_{xx})_i = E\varepsilon_i + \eta^* \frac{\partial \varepsilon_i}{\partial t}, \qquad i=1,2, \tag{1}$$

where E is the Young's modulus and η^* the viscosity of dashpot, $(\sigma_{xx})_1$ and $(\sigma_{xx})_2$ are the axial stresses in the spans $0 < x < x_m$ and $x_m < x < L$, respectively and ε_1 and ε_2 represent the corresponding axial strains.

The bending moment, $(MT)_i$ $(i = 1,2)$, in each span is related to the corresponding displacement field, w_i $(i = 1,2)$, by

$$(MT)_i = EI\frac{\partial^2 w_i}{\partial x^2} + \eta^* I \frac{\partial^3 w_i}{\partial x^2 \partial t}, \qquad i = 1,2. \tag{2}$$

It should be noted that for a slender beam (for example with $I/AL^2 < 0.001$), the linear moment–curvature relationship is sufficiently accurate [17].

For a simply-supported beam, applying Newton's second law of motion in the y direction to an element of each span, and substituting Eqs. (1) and (2) into the resulting equations yields the following set of equations of motion:

$$\begin{aligned}&\rho A\frac{\partial^2 w_1}{\partial t^2} + EI\frac{\partial^4 w_1}{\partial x^4} + \eta^* I\frac{\partial^5 w_1}{\partial x^4 \partial t}\\ &= \frac{3}{2}EA\frac{\partial^2 w_1}{\partial x^2}(\frac{\partial^2 w_1}{\partial x})^2 + \eta^* A\left[\frac{\partial^3 w_1}{\partial x^2 \partial t}\left(\frac{\partial w_1}{\partial x}\right)^2\right.\\ &\left. + 2\frac{\partial^2 w_1}{\partial x^2}\frac{\partial w_1}{\partial x}\frac{\partial^2 w_1}{\partial x \partial t}\right],\end{aligned} \tag{3a}$$

$$\begin{aligned}&\rho A\frac{\partial^2 w_2}{\partial t^2} + EI\frac{\partial^4 w_2}{\partial x^4} + \eta^* I\frac{\partial^5 w_2}{\partial x^4 \partial t}\\ &= \frac{3}{2}EA\frac{\partial^2 w_2}{\partial x^2}(\frac{\partial^2 w_2}{\partial x})^2 + \eta^* A\left[\frac{\partial^3 w_2}{\partial x^2 \partial t}\left(\frac{\partial w_2}{\partial x}\right)^2\right.\\ &\left. + 2\frac{\partial^2 w_2}{\partial x^2}\frac{\partial w_2}{\partial x}\frac{\partial^2 w_2}{\partial x \partial t}\right],\end{aligned} \tag{3b}$$

$$w_1\big|_{x=0} = 0, \quad EI\left.\frac{\partial^2 w_1}{\partial x^2}\right|_{x=0} = 0, \tag{3c}$$

$$w_2\big|_{x=L} = 0, \quad EI\left.\frac{\partial^2 w_2}{\partial x^2}\right|_{x=L} = 0, \tag{3d}$$

$$\begin{aligned}&w_1\big|_{x=x_m} = w_2\big|_{x=x_m}, \quad \left.\frac{\partial w_1}{\partial x}\right|_{x=x_m} = \left.\frac{\partial w_2}{\partial x}\right|_{x=x_m},\\ &EI\left.\frac{\partial^2 w_1}{\partial x^2}\right|_{x=x_m} + \eta^* I\left.\frac{\partial^3 w_1}{\partial x^2 \partial t}\right|_{x=x_m}\\ &= EI\left.\frac{\partial^2 w_2}{\partial x^2}\right|_{x=x_m} + \eta^* I\left.\frac{\partial^3 w_2}{\partial x^2 \partial t}\right|_{x=x_m},\\ &EI\left[\left.\frac{\partial^3 w_1}{\partial x^3}\right|_{x=x_m} - \left.\frac{\partial^3 w_2}{\partial x^3}\right|_{x=x_m}\right]\\ &+\eta^* I\left[\left.\frac{\partial^4 w_1}{\partial x^3 \partial t}\right|_{x=x_m} - \left.\frac{\partial^4 w_2}{\partial x^3 \partial t}\right|_{x=x_m}\right] - M\left.\frac{\partial^2 w_1}{\partial t^2}\right|_{x=x_m} +\\ &\eta^* A\left.\frac{\partial w_1}{\partial x}\right|_{x=x_m}\left[\left.\left(\frac{\partial^2 w_2}{\partial x \partial t}\frac{\partial w_2}{\partial x}\right)\right|_{x=x_m} - \left.\left(\frac{\partial^2 w_1}{\partial x \partial t}\frac{\partial w_1}{\partial x}\right)\right|_{x=x_m}\right] = 0.\end{aligned} \tag{3e}$$

By introducing the dimensionless parameters in the following form:

[2] Although Hamilton's principle seems more appropriate, since it renders additionally the appropriate boundary conditions, it is difficult to employ it here, due to the presence of energy dissipation Kelvin-Voigt type mechanism.

$$\xi=\frac{x}{L},\xi_m=\frac{x_m}{L},\hat{\eta}=\frac{w_1}{r},\hat{\zeta}=\frac{w_2}{r},\tau=\frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}t,$$
$$\varphi=\sqrt{\frac{\eta^{*2}I}{L^4E\rho A\varepsilon^2}},\ \Gamma=\frac{M}{\rho AL}, \tag{4}$$

where r is the radius of gyration and ε illustrates the terms, which are small compared to the other terms, the dimensionless equation of the transverse motion and the corresponding boundary and compatibility conditions become

$$\frac{\partial^2\hat{\eta}}{\partial\tau^2}+\frac{\partial^4\hat{\eta}}{\partial\xi^4}+\varepsilon\varphi\frac{\partial^5\hat{\eta}}{\partial\xi^4\partial\tau}
=\frac{3}{2}\frac{\partial^2\hat{\eta}}{\partial\xi^2}\left(\frac{\partial\hat{\eta}}{\partial\xi}\right)^2+\varepsilon\varphi\left[\frac{\partial^3\hat{\eta}}{\partial\xi^2\partial\tau}\left(\frac{\partial\hat{\eta}}{\partial\xi}\right)^2
+2\frac{\partial^2\hat{\eta}}{\partial\xi^2}\frac{\partial\hat{\eta}}{\partial\xi}\frac{\partial^2\hat{\eta}}{\partial\xi\partial\tau}\right]+\hat{F}_1\cos\Omega\tau \tag{5a}$$

$$\frac{\partial^2\hat{\zeta}}{\partial\tau^2}+\frac{\partial^4\hat{\zeta}}{\partial\xi^4}+\varepsilon\varphi\frac{\partial^5\hat{\zeta}}{\partial\xi^4\partial\tau}
=\frac{3}{2}\frac{\partial^2\hat{\zeta}}{\partial\xi^2}\left(\frac{\partial\hat{\zeta}}{\partial\xi}\right)^2+\varepsilon\varphi\left[\frac{\partial^3\hat{\zeta}}{\partial\xi^2\partial\tau}\left(\frac{\partial\hat{\zeta}}{\partial\xi}\right)^2
+2\frac{\partial^2\hat{\zeta}}{\partial\xi^2}\frac{\partial\hat{\zeta}}{\partial\xi}\frac{\partial^2\hat{\zeta}}{\partial\xi\partial\tau}\right]+\hat{F}_2\cos\Omega\tau \tag{5b}$$

$$\hat{\eta}\big|_{\xi=0}=0,\ \left.\frac{\partial^2\hat{\eta}}{\partial\xi^2}\right|_{\xi=0}=0, \tag{5c}$$

$$\hat{\zeta}\big|_{\xi=1}=0,\ \left.\frac{\partial^2\hat{\zeta}}{\partial\xi^2}\right|_{\xi=1}=0, \tag{5d}$$

$$\hat{\eta}\big|_{\xi=\xi_m}=\hat{\zeta}\big|_{\xi=\xi_m},\ \left.\frac{\partial\hat{\eta}}{\partial\xi}\right|_{\xi=\xi_m}=\left.\frac{\partial\hat{\zeta}}{\partial\xi}\right|_{\xi=\xi_m},$$
$$\left.\frac{\partial^2\hat{\eta}}{\partial\xi^2}\right|_{\xi=\xi_m}+\varepsilon\varphi\left.\frac{\partial^3\eta}{\partial\xi^2\partial\tau}\right|_{\xi=\xi_m}
=\left.\frac{\partial^2\hat{\zeta}}{\partial\xi^2}\right|_{\xi=\xi_m}+\varepsilon\varphi\left.\frac{\partial^3\zeta}{\partial\xi^2\partial\tau}\right|_{\xi=\xi_m},$$
$$\left.\frac{\partial^3\hat{\eta}}{\partial\xi^3}\right|_{\xi=\xi_m}-\left.\frac{\partial^3\hat{\zeta}}{\partial\xi^3}\right|_{\xi=\xi_m}
+\varepsilon\varphi\left[\left.\frac{\partial^4\hat{\eta}}{\partial\xi^3\partial\tau}\right|_{\xi=\xi_m}-\left.\frac{\partial^4\hat{\zeta}}{\partial\xi^3\partial\tau}\right|_{\xi=\xi_m}\right]-\Gamma\frac{\partial^2\eta}{\partial\tau^2}$$
$$+\varepsilon\varphi\left.\frac{\partial\hat{\eta}}{\partial\xi}\right|_{\xi=\xi_m}\left[\left.\left(\frac{\partial^2\hat{\zeta}}{\partial\xi\partial\tau}\frac{\partial\zeta}{\partial\xi}\right)\right|_{\xi=\xi_m}-\left.\left(\frac{\partial^2\hat{\eta}}{\partial\xi\partial\tau}\frac{\partial\eta}{\partial\xi}\right)\right|_{\xi=\xi_m}\right]=0. \tag{5e}$$

In Eqs. (5a) and (5b), we assume that the whole structure is subjected to dimensionless distributed harmonic force, $\hat{F}_i\cos\Omega\tau$, i= 1, 2. Moreover, as seen in Eq. (5e), the effect of the point mass is *not* considered *small*; this results in the presence of terms with Γ in the equations of order one (see Eq. (7d)).

3. The method of multiple timescales

In this section, the method of multiple timescales, an approximate analytic technique [19-25], is employed to study the dynamics of the system. In this method, expansions for the displacements are sought in the following form [26-33]:

$$\eta(\xi,\tau;\varepsilon)=\eta_0(\xi,T_0,T_1)+\varepsilon\eta_1(\xi,T_0,T_1)+O(\varepsilon^2) \tag{6a}$$
$$\zeta(\xi,\tau;\varepsilon)=\zeta_0(\xi,T_0,T_1)+\varepsilon\zeta_1(\xi,T_0,T_1)+O(\varepsilon^2) \tag{6b}$$

where η_0, η_1, ζ_0, and ζ_1 are the functions of order of magnitude one, $T_0=\tau$ is the fast timescale and $T_1=\varepsilon\tau$ the slow one, and $\varepsilon<<1$; $O(\varepsilon^2)$ denotes terms of order of magnitude ε^2 and smaller.

Employing the transformations $\hat{\eta}=\sqrt{\varepsilon}\eta$, $\hat{\zeta}=\sqrt{\varepsilon}\zeta$, $\hat{F}_1=\varepsilon^{3/2}F_1$ and $\hat{F}_2=\varepsilon^{3/2}F_2$ in Eqs. (5a)-(5e) and using Eqs. (6a), (6b) as well as balancing terms of like powers of ε in the resulting equation gives the following set of equations:

$O(\varepsilon^0)$:

$$\frac{\partial^2\eta_0}{\partial T_0^{\,2}}+\frac{\partial^4\eta_0}{\partial\xi^4}=0,\quad \frac{\partial^2\zeta_0}{\partial T_0^{\,2}}+\frac{\partial^4\zeta_0}{\partial\xi^4}=0, \tag{7a}$$

$$\eta_0\big|_{\xi=0}=0,\ \frac{\partial^2\eta_0}{\partial\zeta_0^{\,2}}\Big|_{\xi=0}=0, \tag{7b}$$

$$\zeta_0\big|_{\xi=1}=0,\ \frac{\partial^2\zeta_0}{\partial\zeta_0^{\,2}}\Big|_{\xi=1}=0, \tag{7c}$$

$$\eta_0\big|_{\xi=\xi_m}=\zeta_0\big|_{\xi=\xi_m},\ \frac{\partial\eta_0}{\partial\xi}\Big|_{\xi=\xi_m}=\frac{\partial\zeta_0}{\partial\xi}\Big|_{\xi=\xi_m},$$
$$\frac{\partial^2\eta_0}{\partial\xi^2}\Big|_{\xi=\xi_m}=\frac{\partial^2\zeta_0}{\partial\xi^2}\Big|_{\xi=\xi_m},$$
$$\frac{\partial^3\eta_0}{\partial\xi^3}\Big|_{\xi=\xi_m}-\frac{\partial^3\zeta_0}{\partial\xi^3}\Big|_{\xi=\xi_m}-\Gamma\frac{\partial^2\eta_0}{\partial T_0^2}\Big|_{\xi=\xi_m}=0, \tag{7d}$$

$O(\varepsilon)$:

$$\frac{\partial^2\eta_1}{\partial T_0^{\,2}}+\frac{\partial^4\eta_1}{\partial\xi^4}=-2\frac{\partial^2\eta_0}{\partial T_0\partial T_1}-\varphi\frac{\partial^5\eta_0}{\partial\xi^4\partial T_0}
+\frac{3}{2}\frac{\partial^2\eta_0}{\partial\xi^2}\left(\frac{\partial\eta_0}{\partial\xi}\right)^2+F_1\cos\Omega T_0$$
$$\frac{\partial^2\zeta_1}{\partial T_0^{\,2}}+\frac{\partial^4\zeta_1}{\partial\xi^4}=-2\frac{\partial^2\zeta_0}{\partial T_0\partial T_1}-\varphi\frac{\partial^5\zeta_0}{\partial\xi^4\partial T_0}
+\frac{3}{2}\frac{\partial^2\zeta_0}{\partial\xi^2}\left(\frac{\partial\zeta_0}{\partial\xi}\right)^2+F_2\cos\Omega T_0 \tag{8a}$$

$$\eta_1\big|_{\xi=0}=0,\ \frac{\partial^2\eta_1}{\partial\xi^2}\Big|_{\xi=0}=0, \tag{8b}$$

$$\zeta_1\big|_{\xi=1}=0,\ \frac{\partial^2\zeta_1}{\partial\xi^2}\Big|_{\xi=1}=0, \tag{8c}$$

$$\eta_1\big|_{\xi=\xi_m}=\zeta_1\big|_{\xi=\xi_m},\frac{\partial\eta_1}{\partial\xi}\Big|_{\xi=\xi_m}=\frac{\partial\zeta_1}{\partial\xi}\Big|_{\xi=\xi_m},$$
$$\frac{\partial^2\eta_1}{\partial\xi^2}\Big|_{\xi=\xi_m}-\frac{\partial^2\zeta_1}{\partial\xi^2}\Big|_{\xi=\xi_m}=\varphi\left[\frac{\partial^3\zeta_0}{\partial\xi^2\partial T_0}\Big|_{\xi=\xi_m}-\frac{\partial^3\eta_0}{\partial\xi^2\partial T_0}\Big|_{\xi=\xi_m}\right],$$
$$\frac{\partial^3\eta_1}{\partial\xi^3}\Big|_{\xi=\xi_m}-\frac{\partial^3\zeta_1}{\partial\xi^3}\Big|_{\xi=\xi_m}+\varphi\left[\frac{\partial^4\eta_0}{\partial\xi^3\partial T_0}\Big|_{\xi=\xi_m}-\frac{\partial^4\zeta_0}{\partial\xi^3\partial T_0}\Big|_{\xi=\xi_m}\right]$$
$$-\Gamma\left[\frac{\partial^2\eta_1}{\partial T_0^2}\Big|_{\xi=\xi_m}+2\frac{\partial^2\eta_0}{\partial T_0\partial T_1}\Big|_{\xi=\xi_m}\right]=0. \tag{8d}$$

The general solution for Eqs. (7a)-(7d) may be expressed as a series expansion in terms of the slow timescale amplitude $X_n(T_1)$, the nth natural frequency $\omega_n=\beta_n^2$, and the nth mode functions, $Y_{1n}(\xi)$ and $Y_{2n}(\xi)$, as follows:

$$\eta_0(\xi,T_0,T_1)=\sum_{n=1}^{\infty}\left[X_n(T_1)e^{i\omega_nT_0}+\bar{X}_n(T_1)e^{-i\omega_nT_0}\right]Y_{1n}(\xi), \tag{9a}$$

$$\zeta_0(\xi,T_0,T_1)=\sum_{n=1}^{\infty}\left[X_n(T_1)e^{i\omega_nT_0}+\bar{X}_n(T_1)e^{-i\omega_nT_0}\right]Y_{2n}(\xi). \tag{9b}$$

Substituting Eqs. (9a), (9b) into Eqs. (7a)-(7d) gives

$$\frac{d^4Y_{1n}}{d\xi^4}-\omega_n^2Y_{1n}=0, \tag{10a}$$
$$\frac{d^4Y_{2n}}{d\xi^4}-\omega_n^2Y_{2n}=0, \tag{10b}$$
$$Y_{1n}\big|_{\xi=0}=\frac{d^2Y_{1n}}{d\xi^2}\bigg|_{\xi=0}=0, \tag{10c}$$
$$Y_{2n}\big|_{\xi=1}=\frac{d^2Y_{2n}}{d\xi^2}\bigg|_{\xi=1}=0, \tag{10d}$$
$$Y_{1n}\big|_{\xi=\xi_m}=Y_{2n}\big|_{\xi=\xi_m},\ \frac{dY_{1n}}{d\xi}\bigg|_{\xi=\xi_m}=\frac{dY_{2n}}{d\xi}\bigg|_{\xi=\xi_m},$$
$$\frac{d^2Y_{1n}}{d\xi^2}\bigg|_{\xi=\xi_m}=\frac{d^2Y_{2n}}{d\xi^2}\bigg|_{\xi=\xi_m}, \tag{10e}$$
$$\frac{d^3Y_{1n}}{d\xi^3}\bigg|_{\xi=\xi_m}-\frac{d^3Y_{2n}}{d\xi^3}\bigg|_{\xi=\xi_m}+\Gamma\omega_n^2Y_{1n}\big|_{\xi=\xi_m}=0.$$

Solving Eqs. (10a)-(10e) gives

$$Y_{1n}(\xi)=c_n\left\{\frac{\left(-\sinh(\beta_n)\cosh(\beta_n\xi_m)+\cosh(\beta_n)\sinh(\beta_n\xi_m)\right)}{\sinh(\beta_n)\sin(\beta_n\xi_m)}\sinh(\beta_n\xi)+\frac{\left(-\cos(\beta_n)\sin(\beta_n\xi_m)+\cos(\beta_n\xi_m)\sin(\beta_n)\right)}{\sin(\beta_n\xi_m)\sin(\beta_n)}\sin(\beta_n\xi)\right\}, \tag{11a}$$

$$Y_{2n}(\xi)=c_n\left\{-\frac{\sinh(\beta_n\xi_m)}{\sin(\beta_n\xi_m)}\cosh(\beta_n\xi)+\frac{\cosh(\beta_n)\sinh(\beta_n\xi_m)}{\sinh(\beta_n)\sin(\beta_n\xi_m)}\sinh(\beta_n\xi)-\frac{\cos(\beta_n)}{\sin(\beta_n)}\sin(\beta_n\xi)+\cos(\beta_n\xi)\right\}. \tag{11b}$$

Inserting Eqs. (11a) and (11b) into Eqs. (10c)-(10e) and satisfying the requirement for non-trivial solutions of $Y_{1n}(\xi)$ and $Y_{2n}(\xi)$ yields a scalar nonlinear algebraic equation in terms of $\beta_n=\sqrt{\omega_n}$, which is solved numerically to obtain the linear natural frequencies, ω_n, of the system.

Eqs. (9a) and (9b) are a general solutions for Eqs. (7a)-(7d), including internal resonances. However, since this article considers the system only under resonant excitation at the nth natural frequency, ω_n, it is sufficient to retain only the nth mode. Inserting this nth mode solution along with $\Omega=\omega_n+\varepsilon\sigma$ (σ is a detuning parameter) into Eqs. (8a)-(8d) and expressing the trigonometric functions in exponential forms yields

$$O(\varepsilon):\ \frac{\partial^2\eta_1}{\partial T_0^2}+\frac{\partial^4\eta_1}{\partial\xi^4}=\left[-2i\omega_n\frac{dX_n}{dT_1}Y_{1n}-\varphi i\omega_nX_n\frac{d^4Y_{1n}}{d\xi^4}+\frac{9}{2}X_n^2\bar{X}_n\frac{d^2Y_{1n}}{d\xi^2}\left(\frac{dY_{1n}}{d\xi}\right)^2+\frac{1}{2}F_1e^{i\sigma T_1}\right]e^{i\omega_nT_0}+cc+NST, \tag{12a}$$

$$O(\varepsilon):\ \frac{\partial^2\zeta_1}{\partial T_0^2}+\frac{\partial^4\zeta_1}{\partial\xi^4}=\left[-2i\omega_n\frac{dX_n}{dT_1}Y_{2n}-\varphi i\omega_nX_n\frac{d^4Y_{2n}}{d\xi^4}+\frac{9}{2}X_n^2\bar{X}_n\frac{d^2Y_{2n}}{d\xi^2}\left(\frac{dY_{2n}}{d\xi}\right)^2+\frac{1}{2}F_2e^{i\sigma T_1}\right]e^{i\omega_nT_0}+cc+NST, \tag{12b}$$

$$\eta_1\big|_{\xi=0}=\frac{\partial^2\eta_1}{\partial\xi^2}\bigg|_{\xi=0}=0, \tag{12c}$$

$$\zeta_1\big|_{\xi=1}=\frac{\partial^2\zeta_1}{\partial\xi^2}\bigg|_{\xi=1}=0, \tag{12d}$$

$$\eta_1\big|_{\xi=\xi_m}=\zeta_1\big|_{\xi=\xi_m},\ \frac{\partial\eta_1}{\partial\xi}\bigg|_{\xi=\xi_m}=\frac{\partial\zeta_1}{\partial\xi}\bigg|_{\xi=\xi_m},$$
$$\frac{\partial^2\eta_1}{\partial\xi^2}\bigg|_{\xi=\xi_m}=\frac{\partial^2\zeta_1}{\partial\xi^2}\bigg|_{\xi=\xi_m},$$

$$\frac{\partial^3\eta_1}{\partial\xi^3}\bigg|_{\xi=\xi_m}-\frac{\partial^3\zeta_1}{\partial\xi^3}\bigg|_{\xi=\xi_m}-\Gamma\frac{d^2\eta_1}{dT_0^2}\bigg|_{\xi=\xi_m}=\left[\varphi i\omega_nX_n\left(\frac{d^3Y_{2n}}{d\xi^3}\bigg|_{\xi=\xi_m}-\frac{d^3Y_{1n}}{d\xi^3}\bigg|_{\xi=\xi_m}\right)+2\Gamma i\omega_n\frac{dX_n}{dT_1}Y_{1n}\bigg|_{\xi=\xi_m}\right]e^{i\omega_nT_0}+cc, \tag{12e}$$

where *cc* represents the complex conjugate of the preceding terms on the right-hand side of the equation and *NST* denotes non-secular terms.

Fulfilling the *solvability condition* [31] for Eqs. (12a)-(12e) gives the following equation, which is an ordinary differential equation in terms of T_1 :

$$\gamma_{1n}X_n^2\bar{X}_n + i\gamma_{2n}\frac{dX_n}{dT_1} + i\gamma_{3n}X_n = \frac{1}{2}\gamma_{4n}e^{i\sigma T_1} \tag{13a}$$

in which

$$\gamma_{1n} = -\frac{9}{2}\left[\int_0^{\xi_m} Y_{1n}\frac{d^2Y_{1n}}{d\xi^2}\left(\frac{dY_{1n}}{d\xi}\right)^2 d\xi + \int_{\xi_m}^1 Y_{2n}\frac{d^2Y_{2n}}{d\xi^2}\left(\frac{dY_{2n}}{d\xi}\right)^2 d\xi\right], \tag{13b}$$

$$\gamma_{2n} = 2\Gamma\omega_n Y_{1n}^2\Big|_{\xi=\xi_m} + 2\omega_n\left[\int_0^{\xi_m} Y_{1n}^2 d\xi + \int_{\xi_m}^1 Y_{2n}^2 d\xi\right], \tag{13c}$$

$$\gamma_{3n} = \varphi\omega_n\left[Y_{1n}\Big|_{\xi=\xi_m}\left(\frac{d^3Y_{2n}}{d\xi^3}\bigg|_{\xi=\xi_m} - \frac{d^3Y_{1n}}{d\xi^3}\bigg|_{\xi=\xi_m}\right) + \int_0^{\xi_m} Y_{1n}\frac{d^4Y_{1n}}{d\xi^4}d\xi + \int_{\xi_m}^1 Y_{2n}\frac{d^4Y_{2n}}{d\xi^4}d\xi\right], \tag{13d}$$

$$\gamma_{4n} = F_1\int_0^{\xi_m} Y_{1n}d\xi + F_2\int_{\xi_m}^1 Y_{2n}d\xi. \tag{13e}$$

Considering X_n as

$$X_n(T_1) = \frac{1}{2}a_n(T_1)e^{i\beta_n(T_1)} \tag{14}$$

where $a_n(T_1)$ and $\beta_n(T_1)$ represent the real-valued functions of slow timescale, and substituting it into Eq. (13a), then separating the real and imaginary parts as well as writing $\theta_n(T_1) = \sigma T_1 - \beta_n(T_1)$ gives

$$\frac{1}{4}\gamma_{1n}a_n^3 - \gamma_{2n}a_n\frac{d\beta_n}{dT_1} - \gamma_{4n}\cos\theta_n = 0\,, \tag{15a}$$

$$\gamma_{2n}\frac{da_n}{dT_1} + \gamma_{3n}a_n - \gamma_{4n}\sin\theta_n = 0\,. \tag{15b}$$

Considering forced vibrations, if the amplitude a_n and phase do not change with time in Eqs. (15a) and (15b), the steady-state response is obtained. Fulfilling this requirement and solving the resulting equations for σ gives the following frequency-response equations:

$$\sigma_1 = \frac{\frac{1}{2}\gamma_{1n}\gamma_{2n}a_n^4 - \sqrt{\left(\frac{1}{2}\gamma_{1n}\gamma_{2n}a_n^4\right)^2 - 4\left(\gamma_{2n}a_n\right)^2\left[\left(\frac{1}{4}\gamma_{1n}a_n^3\right)^2 + \left(\gamma_{3n}a_n\right)^2 - \left(\gamma_{4n}\right)^2\right]}}{2\left(\gamma_{2n}a_n\right)^2}, \tag{16a}$$

$$\sigma_2 = \frac{\frac{1}{2}\gamma_{1n}\gamma_{2n}a_n^4 + \sqrt{\left(\frac{1}{2}\gamma_{1n}\gamma_{2n}a_n^4\right)^2 - 4\left(\gamma_{2n}a_n\right)^2\left[\left(\frac{1}{4}\gamma_{1n}a_n^3\right)^2 + \left(\gamma_{3n}a_n\right)^2 - \left(\gamma_{4n}\right)^2\right]}}{2\left(\gamma_{2n}a_n\right)^2}. \tag{16b}$$

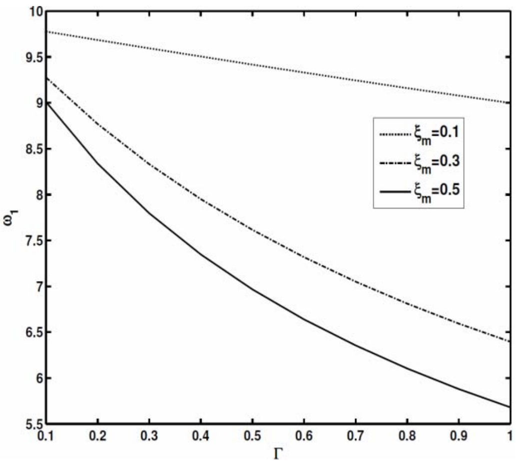

Fig. 3. The first linear natural frequency of the system as a function of dimensionless parameter Γ , for several mass locations.

For the case of free vibrations, i.e. $\gamma_{4n} = 0$, solving Eq. (15b) for a_n results in

$$a_n(\varepsilon t) = a_{0n}e^{-\frac{\gamma_{3n}}{\gamma_{2n}}\varepsilon t} \tag{17}$$

where a_{0n} is a constant. Substituting this into Eq. (15b) and simplifying the resulting equation gives the nonlinear natural frequency as follows:

$$\left(\omega_{NL}\right)_n = \omega_n + \frac{1}{4}\varepsilon\frac{\gamma_{1n}}{\gamma_{2n}}a_{0n}^2 e^{-2\frac{\gamma_{3n}}{\gamma_{2n}}\varepsilon t}\,. \tag{18}$$

The mode function of each span, and the time-dependent nonlinear amplitude and frequency have been determined in Eqs. (11a), (11b), (17) and (18), respectively. Inserting these into Eq. (9a) and (9b) gives the first-order, free nonlinear response of the system.

4. Numerical parametric study

The effects of system parameters such as the mass value and location, viscosity coefficient and forcing amplitude on the linear and nonlinear natural frequencies, vibration responses, and frequency-response curves of the system are investigated via a numerical parametric study.

The first case examined is the effect of dimensionless mass parameter Γ on the first three linear natural frequencies of the system. Fig. 3 shows the effect of mass value on the first natural frequency for different mass locations. In fact, when the mass is located at the middle of the beam, increasing the mass value results in smaller natural frequencies compared to other mass locations. However, as depicted in Fig. 4, adding the mass value at the middle has no effect on the second natural frequency of the system. Moreover, it is evident from Fig. 5 that by increasing the amount of mass from $\Gamma = 0.5$ to larger values, the third natural frequency decays significantly for the cases for which the mass is close to either the left or right end of the beam. However, for small mass values

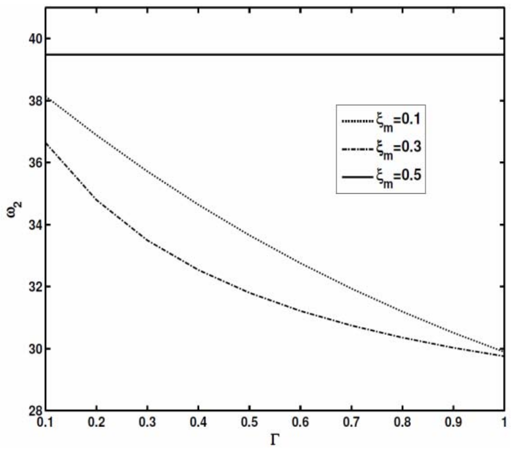

Fig. 4. The second linear natural frequency of the system as a function of dimensionless parameter Γ, for several mass locations.

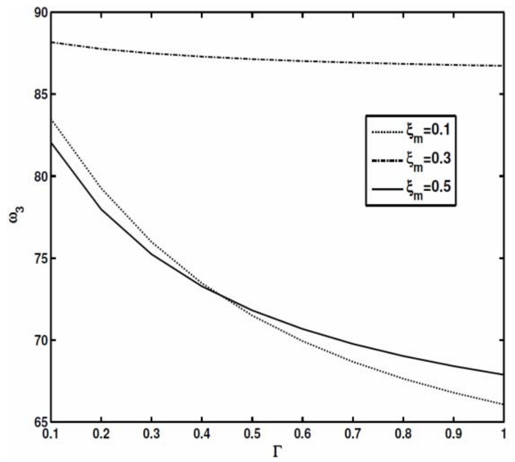

Fig. 5. The third linear natural frequency of the system as a function of dimensionless parameter Γ, for several mass locations.

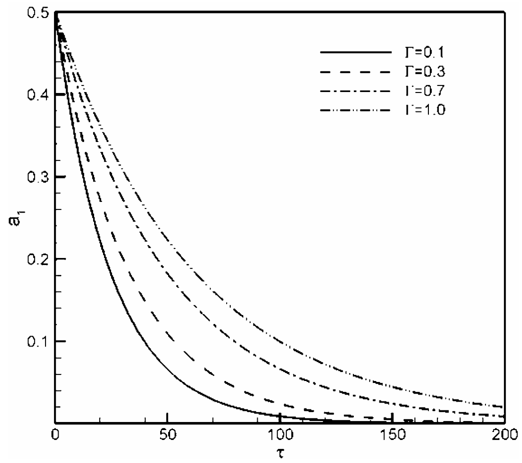

Fig. 6. The amplitude of the first mode of vibration as a function of time for several mass values; $\xi_m = 0.5$, $\varphi = 0.001$.

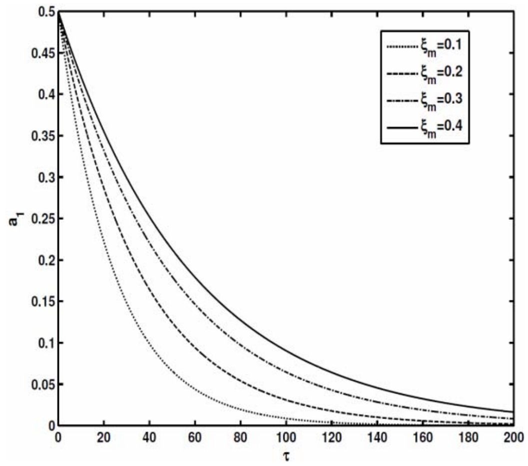

Fig. 7. The amplitude of the first mode of vibration as a function of time for several mass locations; $\Gamma = 1$, $\varphi = 0.001$.

($0.1 < \Gamma < 0.5$), this frequency is smaller when the mass is located in the middle. Fig. 4 shows that the second natural frequency for ξ_m=0.5 is not significantly affected by the mass value. The reason is that the mass is attached to the nodal point (ξ_m=0.5) for the second vibration mode of the simply-supported beam. Moreover, as seen in Fig. 5, for the case with ξ_m=0.3, the third natural frequency is not that sensitive to the mass value. This is because the mass is attached to the vicinity of the nodal point (ξ_m =1/3) for the third vibration mode of the simply-supported beam, and hence has a minor effect on the third natural frequency.

The next aspect considered was the amplitude of the first mode of vibration. The first mode amplitude as a function of time for several mass values, mass locations, and viscosity coefficients are given in Figs. 6, 7 and 8, respectively. Examining these figures reveals that due to the viscosity effect, the amplitude decays with time, and for larger values of Γ and ξ_m (from 0 to 0.5), the amplitude decays more slowly.

The next aspect studied was the influence of the system parameters on the nonlinear natural frequency of the system. The backbone curves are illustrated for several mass values and locations in Figs. 9 and 10, respectively. Also, Figs. 11 and 12 show the variation of the first nonlinear natural frequency with time. Apparently, for the case of heavy point mass at the middle, the first nonlinear natural frequency is the smallest; as seen here, the nonlinear natural frequencies are *time-dependent*.

In order to study the frequency-response characteristics, the primary resonance response of the system is investigated for several mass locations, mass values, forcing amplitudes, and viscosity coefficients in Figs. 13-16. It can be concluded that by either moving the mass from the left-end to the center of the beam or increasing the mass value, weaker hardening nonlinearity is obtainable. In other words, the multi-valued regions become smaller and subsequently, the jump phenomenon is affected. Finally, Fig. 15 shows smaller peak amplitudes for larger viscosity coefficients and Fig. 16 shows a larger response for bigger forcing amplitude.

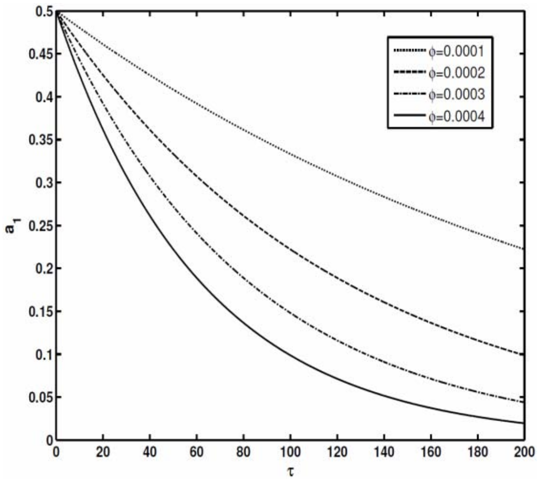

Fig. 8. The amplitude of the first mode of vibration as a function of time for several viscosity coefficients; $\Gamma = 0.1,\ \xi_m = 0.5$.

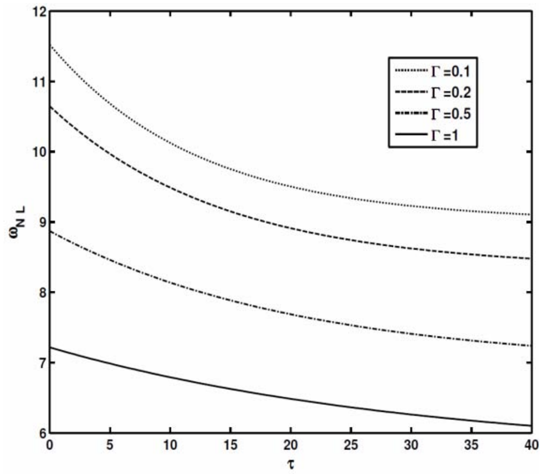

Fig. 11. The first nonlinear natural frequency of the system as a function of time for several mass values; $a_{0n} = 1,\ \xi_{0m} = 0.5,\ \varphi = 0.001$.

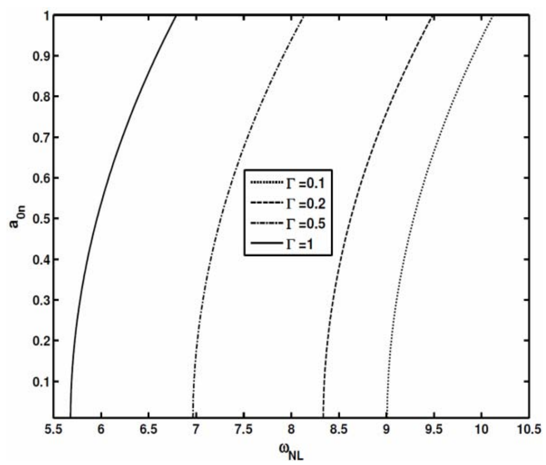

Fig. 9. The backbone curves of the system for several mass values; $\tau = 10,\ \xi_m = 0.5,\ \varphi = 0.001$.

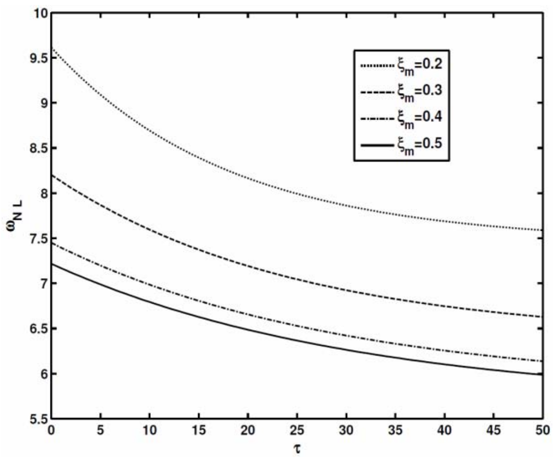

Fig. 12. The first nonlinear natural frequency of the system as a function of time for several mass locations; $a_{0n} = 1,\ \Gamma = 1,\ \varphi = 0.001$.

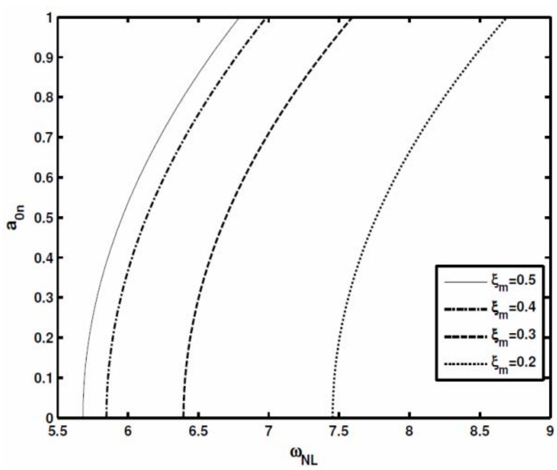

Fig. 10. The backbone curves of the system for several mass locations; $\tau = 10,\ \Gamma = 1,\ \varphi = 0.001$.

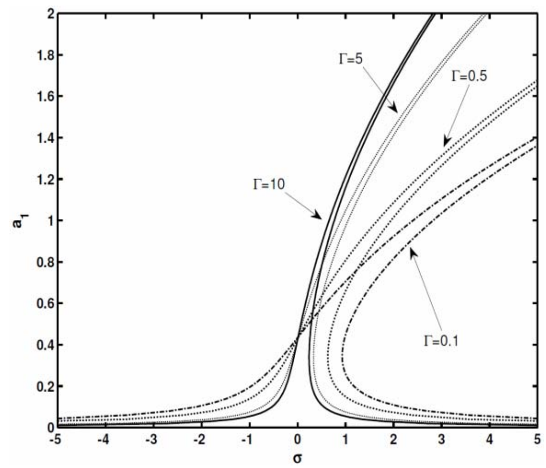

Fig. 13. The frequency-response curve of the system for several mass values; $\xi_m = 0.3,\ \varphi = 0.001,\ F_1 = F_2 = F = 5$.

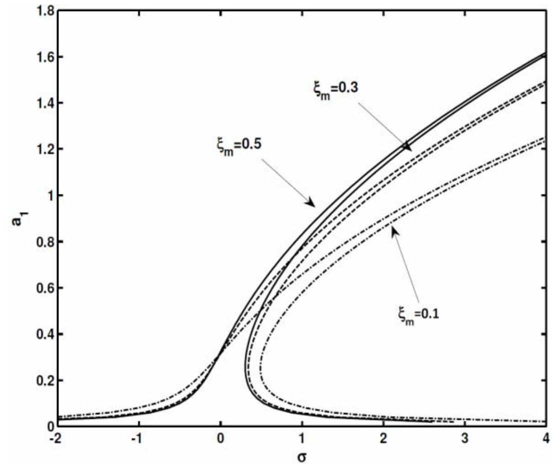

Fig. 14. The frequency-response curve of the system for several values of the mass location; $\Gamma = 1$, $\varphi = 0.001$, $F_1 = F_2 = F = 2$.

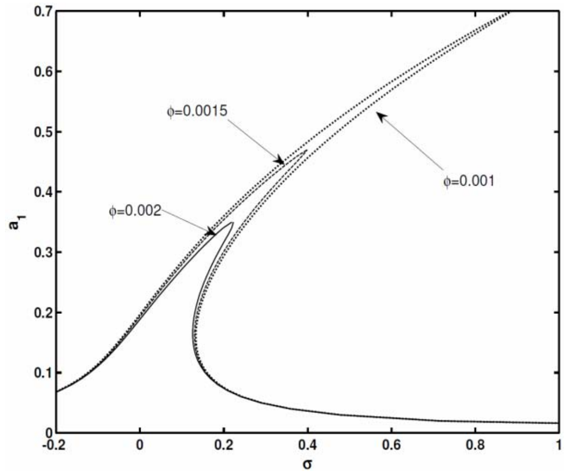

Fig. 15. The frequency-response curve of the system for several values of the viscosity coefficient; $\Gamma = 1$, $\xi_m = 0.3$, $F_1 = F_2 = F = 0.5$.

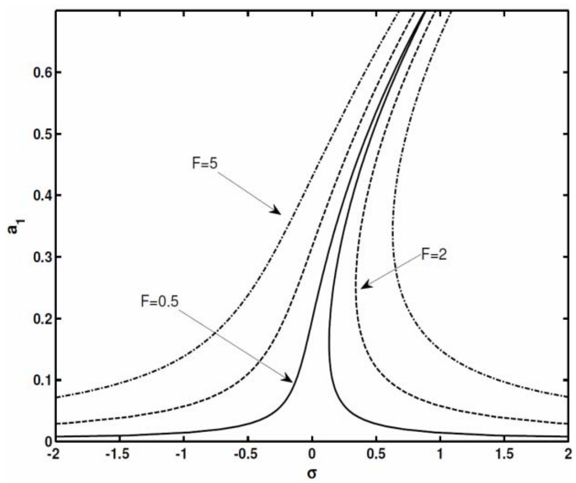

Fig. 16. The frequency-response curve of the system for several values of the forcing amplitude; $\Gamma = 1$, $\xi_m = 0.3$, $\varphi = 0.001$, $F_1 = F_2 = F$.

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5. Conclusions

The equations of motion for a Kelvin-Voigt viscoelastic beam carrying a heavy, intra-span point-mass has been derived using Newton's second law of motion and has been solved via multiple scales perturbation technique. It was shown that the presence of a heavy mass can affect the vibration characteristics of the system and the location of the mass influences the linear as well as nonlinear vibration responses. In fact, it is revealed that due to the viscosity effect, the amplitude of vibration and nonlinear natural frequency decay with time, and for greater mass values and larger values of ξ_m (from 0 to 0.5), the amplitude decays even more slowly. Moreover, for cases where the mass is closer to either left or right end of the beam, stronger hardening behavior is observed. Conversely, for greater mass values, the hardening effect is weakened and the multi-valued region becomes smaller. In conclusion, it can be said that the results indicate the importance of considering energy dissipation models in beams with adornments to better meet the desired performance of these systems in terms of both design and control.

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