



Computer Algebra in Theoretical Physics

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Abstract:

Generally speaking, physicists still experience that computing with paper and pencil is in most cases simpler than computing on a Computer Algebra worksheet. On the other hand, recent developments in the Maple system have implemented most of the mathematical objects and mathematics used in theoretical physics computations, and have dramatically approximated the notation used in the computer to the one used with paper and pencil, diminishing the learning gap and computer-syntax distraction to a strict minimum.

In this talk, the Physics project at Maplesoft is presented and the resulting Physics package is illustrated by tackling problems in classical and quantum mechanics, using tensor and Dirac's Bra-Ket notation, general relativity, including the equivalence problem, and classical field theory, deriving field equations using variational principles.



... and why computer algebra?

We can concentrate more on the ideas instead of on the algebraic manipulations

We can extend results with ease

We can explore the mathematics surrounding a problem

We can share results in a reproducible way

▼ **Representation issues that were preventing the use of computer algebra in Physics**

Notation and related mathematical methods that were missing:

*coordinate free representations for vectors and vectorial differential operators,
covariant tensors distinguished from contravariant tensors, sum rule for tensor contracted (repeated) indices
functional differentiation, spacetime and covariant differential operators
Bras, Kets, projectors and all related to Dirac's notation in Quantum Mechanics*

Inert representations of operations, mathematical functions, and related typesetting were missing:

*inert versus active representations for mathematical operations
hand-like style for entering computations, and computationally active output with textbook-like notation*

Key elements of the computational domain of theoretical physics were missing:

*product and differentiation handling commutative, anticommutative and noncommutative variables and functions
ability to set custom-defined algebra rules (commutator,*

anticommutator and bracket rules, etc.)
ability to distinguish between generic, unitary and Hermitian
quantum operators

▼ Classical Mechanics

▼ *Inertia tensor for a triatomic molecule

Problem

Determine the Inertia tensor of a triatomic molecule that has the form of an isosceles triangle with two masses m_1 in the extremes of the base and mass m_2 in the third vertex. The distance between the two masses m_1 is equal to a , and the height of the triangle is equal to h .

▼ Solution

> restart, with (Physics, KroneckerDelta) : with (Physics [Vectors]) :

The general formula, where \vec{r}_k is the position of each particle measured in a reference system with the origin at the "center of mass"

> InertiaTensor := Sum (m[k] (Norm(r_[k])² KroneckerDelta[i,j] - Component(r_[k], i) Component(r_[k], j)), k = 1 .. N)

$$InertiaTensor := \sum_{k=1}^N m_k \left(\|\vec{r}_k\|^2 \delta_{i,j} - (\vec{r}_k)_i (\vec{r}_k)_j \right) \quad (1)$$

To have a Matrix representation of this inertia tensor, create an indexing function

> IT := unapply(InertiaTensor, i, j)

$$IT := (i, j) \mapsto \sum_{k=1}^N m_k \left(\|\vec{r}_k\|^2 \delta_{i,j} - (\vec{r}_k)_i (\vec{r}_k)_j \right) \quad (2)$$

In this problem there are 3 particles

> N := 3

$$N := 3 \quad (3)$$

The matrix representation of the inertia tensor

> IT_Matrix := Matrix(3, IT)

IT_Matrix := (4)

$$\left[\left[\sum_{k=1}^3 m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_1^2 \right), \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_2 \right), \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_3 \right) \right], \right.$$

$$\left[\begin{array}{l} \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_2 \right), \sum_{k=1}^3 m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_2^2 \right), \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_2 (\vec{r}_k)_3 \right) \\ \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_1 (\vec{r}_k)_3 \right), \sum_{k=1}^3 \left(-m_k (\vec{r}_k)_2 (\vec{r}_k)_3 \right), \sum_{k=1}^3 m_k \left(\|\vec{r}_k\|^2 - (\vec{r}_k)_3^2 \right) \end{array} \right]$$

The vectors \vec{r}_k (position from the Center of Mass) are related to \vec{R}_k (position in the arbitrarily chosen system of references) and \vec{R}_{CM} (position of the Center of Mass in the arbitrarily chosen system) by

$$\begin{aligned} > \text{position} := r_ [k] = R_ [k] - R_ [CM]; \\ & \text{position} := \vec{r}_k = \vec{R}_k - \vec{R}_{CM} \end{aligned} \quad (5)$$

Where by definition the "center of mass" is

$$\begin{aligned} > R_ [CM] := \text{Sum}(m[j] R_ [j], j = 1 .. N) / \text{Sum}(m[j], j = 1 .. N); \\ & \vec{R}_{CM} := \frac{\sum_{j=1}^3 m_j \vec{R}_j}{\sum_{j=1}^3 m_j} \end{aligned} \quad (6)$$

And that is all the formulation. It is now in the computer. We don't need to waste our brain power keeping these formulas in our minds. Just specify the problem at hands

So, arbitrarily choose now a system of reference, for example with the origin at the middle of the segment connecting the two atoms of equal mass. The position of each mass is then

$$\begin{aligned} > R_ [1] := - \frac{a}{2} _i; \\ & \vec{R}_1 := - \frac{a \hat{i}}{2} \end{aligned} \quad (7)$$

$$\begin{aligned} > R_ [2] := h _k \\ & \vec{R}_2 := h \hat{k} \end{aligned} \quad (8)$$

$$\begin{aligned} > R_ [3] := \frac{a}{2} _i \\ & \vec{R}_3 := \frac{a \hat{i}}{2} \end{aligned} \quad (9)$$

Two masses are equal

$$\begin{aligned} > m[3] := 2 \cdot m[1] \\ & m_3 := 2 m_1 \end{aligned} \quad (10)$$

$$\begin{aligned} > R_ [CM] := \text{value}(R_ [CM]) \\ & \end{aligned} \quad (11)$$

$$\vec{R}_{CM} := \frac{\frac{m_1 a \hat{i}}{2} + m_2 h \hat{k}}{3 m_1 + m_2} \quad (11)$$

The positions of the three particles viewed from the center of mass

> position

$$\vec{r}_k = \vec{R}_k - \frac{\frac{m_1 a \hat{i}}{2} + m_2 h \hat{k}}{3 m_1 + m_2} \quad (12)$$

> position \$ k = 1 .. N

$$\vec{r}_1 = -\frac{a \hat{i}}{2} - \frac{\frac{m_1 a \hat{i}}{2} + m_2 h \hat{k}}{3 m_1 + m_2}, \vec{r}_2 = h \hat{k} - \frac{\frac{m_1 a \hat{i}}{2} + m_2 h \hat{k}}{3 m_1 + m_2}, \vec{r}_3 = \frac{a \hat{i}}{2} - \frac{\frac{m_1 a \hat{i}}{2} + m_2 h \hat{k}}{3 m_1 + m_2} \quad (13)$$

The abstract IT_Matrix at these values of the vectors \vec{r}_k

> IT_answer := simplify(eval(value(IT_Matrix), [(13)]))

IT_answer :=

(14)

$$\begin{bmatrix} \frac{3 h^2 m_1 m_2}{3 m_1 + m_2} & 0 & \frac{m_1 a m_2 h}{6 m_1 + 2 m_2} \\ 0 & \frac{8 a^2 m_1^2 + 3 m_2 (a^2 + 4 h^2) m_1}{12 m_1 + 4 m_2} & 0 \\ \frac{m_1 a m_2 h}{6 m_1 + 2 m_2} & 0 & \frac{m_1 a^2 (8 m_1 + 3 m_2)}{12 m_1 + 4 m_2} \end{bmatrix}$$

Try changing the origin of the arbitrarily chosen system of references, or the value of m_3 in (10) and re-execute the lines after that definition and you see the corresponding answer instantly. Computer algebra allows for these simple recalculations without having to reformulate anything.

>

▼ Quantum mechanics

▼ *The quantum operator components of \vec{L} satisfy $[L_j, L_k]_- = i \epsilon_{j, k, m} L_m$

> restart, with(Physics) : interface(imaginaryunit = i) :

> Setup(spaceindices = lowercaselatin, metric = Euclidean, automaticsimplication = true)

The Euclidean metric in cartesian coordinates

Changing the signature of the tensor spacetime to: + + + +

$$[\text{automaticsimplication} = \text{true}, \text{metric} = \{(1, 1) = 1, (2, 2) = 1, (3, 3) = 1, (4, 4) = 1\}, \text{spaceindices} = \text{lowercaselatin}] \quad (15)$$

Define L , r and p as tensors of the 3-D Euclidean space embedded in

> Define(L, r, p)

Defined objects with tensor properties

$$\{L, p, r, \gamma_\mu, \sigma_\mu, \partial_\mu, g_{\mu, \nu}, \gamma_{a, b}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\} \quad (16)$$

Now set L, p, r as quantum operators and the related **Commutator** rules for the algebra in tensor notation

> Setup($\text{quantumoperators} = \{L, p, r\}$,

{%Commutator($p[j], p[k]$) = 0,

%Commutator($r[j], p[k]$) = i KroneckerDelta[j, k],

%Commutator($r[j], r[k]$) = 0})

$$[\text{algebrarules} = \{[p_j, p_k]_- = 0, [r_j, p_k]_- = i \delta_{j, k}, [r_j, r_k]_- = 0\}, \text{quantumoperators} \quad (17)$$

$$= \{L, p, r\}$$

Verify how these algebra rules work:

> (%Commutator = Commutator)($r[j], p[k]$)

$$[r_j, p_k]_- = i \delta_{j, k} \quad (18)$$

> (%Commutator = Commutator)($r[j], r[k]$)

$$[r_j, r_k]_- = 0 \quad (19)$$

> (%Commutator = Commutator)($p[j], p[k]$);

$$[p_j, p_k]_- = 0 \quad (20)$$

The definition of L_j

> $L[j] = \text{LeviCivita}[j, k, m] r[k] p[m]$

$$L_j = \epsilon_{j, k, m} r_k p_m \quad (21)$$

The rule to be verified:

> %Commutator($L[j], L[k]$) = i LeviCivita[j, k, m] $L[m]$

$$[L_j, L_k]_- = i \epsilon_{j, k, m} L_m \quad (22)$$

Substitute now the operator L_i by its tensor form in terms r_k and p_m in the commutator above

> Library:-SubstituteTensor((21), (22))

$$[\epsilon_{j, a, m} r_a p_m, \epsilon_{k, b, c} r_b p_c]_- = i \epsilon_{m, a, b} r_a p_b \epsilon_{j, k, m} \quad (23)$$

Simplify, all in one go, we expect an identity

> Simplify((23))

$$i (r_j p_k - r_k p_j) = i (r_j p_k - r_k p_j) \quad (24)$$

The same one step at a time, first expand the commutator on the left-hand side

> expand((23))

$$-\epsilon_{a,j,m} \epsilon_{b,c,k} (r_a p_m r_b p_c - r_b p_c r_a p_m) = i \epsilon_{a,b,m} r_a p_b \epsilon_{j,k,m} \quad (25)$$

> Simplify((25))

$$i (r_j p_k - r_k p_j) = i (r_j p_k - r_k p_j) \quad (26)$$

>

▼ *Unitary Operators in Quantum Mechanics

▼ *Eigenvalues of an unitary operator and exponential of Hermitian operators

- Show that the eigenvalues of an unitary operator are all on the unit circle, their modulus is 1.
- Show that an operator $e^{i\lambda H}$ is unitary provided that H is Hermitian ($H = H^\dagger$) and λ is any real parameter.

> restart, with (Physics) : interface(imaginaryunit = i) :

> Setup(unitaryoperators = {U})

$$[\text{unitaryoperators} = \{U\}] \quad (27)$$

If $|U_\epsilon\rangle$ is a normalized eigenvector of U with eigenvalue ϵ

> U·Ket(U, ε) = U · Ket(U, ε)

$$U |U_\epsilon\rangle = \epsilon |U_\epsilon\rangle \quad (28)$$

> Dagger((28))

$$\langle U_\epsilon | U^\dagger = \bar{\epsilon} \langle U_\epsilon | \quad (29)$$

So, to show that the eigenvalues have modulus equal to 1, multiplying sides by sides

> (29) · (28)

$$1 = |\epsilon|^2 \quad (30)$$

To show that, when H is Hermitian, then $V = e^{i\lambda H}$ is unitary,

> Setup(quantumoperators = {V}, hermitianoperators = {H}, realobjects = {λ})

$$[\text{hermitianoperators} = \{H\}, \text{quantumoperators} = \{H, U, V\}, \text{realobjects} = \{\lambda\}] \quad (31)$$

> V = exp(i·λ·H)

$$V = e^{i\lambda H} \quad (32)$$

> Dagger((32))

$$V^\dagger = e^{-i\lambda H} \quad (33)$$

Again multiply sides by sides

> (32) · (33)

$$V V^\dagger = 1 \quad (34)$$

> (33) · (32)

$$V^\dagger V = 1 \quad (35)$$

Therefore, V is unitary

>

▼ *Properties of unitary operators

Consider two set of kets $|a_n\rangle$ and $|b_n\rangle$, each of them constituting a complete orthonormal basis of the same space.

▼ *Verify that $U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k|$, maps one basis to the other, i.e.: $|b_n\rangle = U |a_n\rangle$

> restart; with (Physics) :

Tell the system that $|a_n\rangle$ and $|b_n\rangle$, are complete orthonormal basis

> Setup(quantumoperators = {U},
bracketrules = {%Bracket(Bra(a, m), Ket(a, n)) = KroneckerDelta[m, n],
%Bracket(Bra(b, m), Ket(b, n)) = KroneckerDelta[m, n]})
[bracketrules = {⟨a_m | a_n⟩ = δ_{m, n}, ⟨b_m | b_n⟩ = δ_{m, n}}, quantumoperators = {U}] (36)

> $U = \sum_{k=0}^{\infty} Ket(b, k) Bra(a, k)$

$$U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \quad (37)$$

Apply this operatorial equation to $|a_m\rangle$

> '%. Ket(a, m)'

$$\left(U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \right) \cdot |a_m\rangle \quad (38)$$

> %

$$U \cdot |a_m\rangle = |b_m\rangle \quad (39)$$

▼ *Show that $U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k|$ is unitary

Recalling the expansion of the operator U

> (37)

$$U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \quad (40)$$

> Dagger((37))

$$U^\dagger = \sum_{k=0}^{\infty} \langle a_k| |b_k\rangle \quad (41)$$

Again multiply sides by sides

> '(41) . (37)'

(42)

$$\left(U^\dagger = \sum_{k=0}^{\infty} |a_k\rangle \langle b_k| \right) \cdot \left(U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \right) \quad (42)$$

> %

$$U^\dagger U = \sum_{kl=0}^{\infty} |a_{kl}\rangle \langle a_{kl}| \quad (43)$$

> (37) . (41)

$$U U^\dagger = \sum_{kl=0}^{\infty} |b_{kl}\rangle \langle b_{kl}| \quad (44)$$

and since $|a_n\rangle$ and $|b_n\rangle$ form two complete basis of the same space, the right-hand sides are equal to the identity operator $\mathbb{1}$, and so U is unitary.

>

▼ *Show that the matrix elements of U in the $|a_n\rangle$ and $|b_n\rangle$ basis are equal

Recalling the expansion of the operator U

> (37)

$$U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \quad (45)$$

Compute now the matrix elements of U in the $|a_n\rangle$ and $|b_n\rangle$ basis

> 'Bra(a, n) . (37) . Ket(a, m)'

$$\langle a_n | \cdot \left(U = \sum_{k=0}^{\infty} |b_k\rangle \langle a_k| \right) \cdot | a_m \rangle \quad (46)$$

> %

$$\langle a_n | U | a_m \rangle = \langle a_n | b_m \rangle \quad (47)$$

Likewise

> Bra(b, n) . (37) . Ket(b, m)

$$\langle b_n | U | b_m \rangle = \langle a_n | b_m \rangle \quad (48)$$

>

▼ Show that A and $\mathcal{A} = U A U^\dagger$ have the same spectrum (eigenvalues)

> Setup (redo, quantumoperators = $\{A, \mathcal{A}\}$, unitaryoperators = $\{U\}$)

[quantumoperators = $\{\mathcal{A}, A, U\}$, unitaryoperators = $\{U\}$] (49)

> $U A \text{Dagger}(U) = \mathcal{A}$

$$U A U^\dagger = \mathcal{A} \quad (50)$$

By construction the eigenkets of \mathcal{A} are $|\mathcal{A}_\alpha\rangle = U |A_\alpha\rangle$

> $U \cdot \text{Ket}(A, \alpha) = \text{Ket}(\mathcal{A}, \alpha)$

$$U \cdot |A_\alpha\rangle = |\mathcal{A}_\alpha\rangle \quad (51)$$

> (50) · (51)

$$U A U^\dagger (U \cdot |A_\alpha\rangle) = \mathcal{A} | \mathcal{A}_\alpha \rangle \quad (52)$$

The left-hand side can be rewritten performing the product

> lhs((52)) = eval(lhs((52)), `*` = `.`)

$$U A U^\dagger (U \cdot |A_\alpha\rangle) = \alpha (U \cdot |A_\alpha\rangle) \quad (53)$$

> subs((53), (52))

$$\alpha (U \cdot |A_\alpha\rangle) = \mathcal{A} | \mathcal{A}_\alpha \rangle \quad (54)$$

> subs((51), (54))

$$\alpha | \mathcal{A}_\alpha \rangle = \mathcal{A} | \mathcal{A}_\alpha \rangle \quad (55)$$

>

In conclusion, after an unitary transformation, an eigenvector of the initial operator is an eigenvector of the new operator with the same eigenvalue.

▼ Schrödinger equation and unitary transform

Consider a ket $|\psi_t\rangle$ that solves the time-dependant Schrödinger equation:

$$i \hbar \frac{\partial}{\partial t} |\psi_t\rangle = H(t) |\psi_t\rangle$$

and consider

$$|\phi_t\rangle = U(t) |\psi_t\rangle,$$

where $U(t)$ is a unitary operator.

Does $|\phi_t\rangle$ evolves according a Schrödinger equation

$$i \cdot \hbar \frac{\partial}{\partial t} |\phi_t\rangle = \mathcal{H}(t) |\phi_t\rangle$$

and if yes, which is the expression of $\mathcal{H}(t)$?

▼ Solution

> restart, with (Physics) : interface(imaginaryunit = i) :

> Setup(automaticsimplication = true, mathematicalnotation = true, quantumoperators = {H}, hermitianoperators = {H}, unitaryoperators = {U}, realobjects = {t, h})

[automaticsimplication = true, hermitianoperators = {H}, mathematicalnotation = true, quantumoperators = {H, U}, realobjects = {h, t}, unitaryoperators = {U}] (56)

> CompactDisplay((U, H, H)(t))

$U(t)$ will now be displayed as U

$H(t)$ will now be displayed as H

$$\mathcal{H}(t) \text{ will now be displayed as } \mathcal{H} \quad (57)$$

$$\begin{aligned} > \text{Ket}(\phi, t) = U(t) \cdot \text{Ket}(\psi, t) \\ & \quad \quad \quad |\phi_t\rangle = U |\psi_t\rangle \end{aligned} \quad (58)$$

Compute now the evolution of $|\phi_t\rangle$

$$\begin{aligned} > i \cdot \hbar \cdot \text{diff}((58), t) \\ & \quad \quad \quad i \hbar |\phi_t\rangle_t = i \hbar (U_t |\psi_t\rangle + U |\psi_t\rangle_t) \end{aligned} \quad (59)$$

Simplify this equation taking into account Schrödinger's equation for ψ :

$$\begin{aligned} > i \cdot \hbar \frac{\partial}{\partial t} \text{Ket}(\psi, t) = H(t) \text{Ket}(\psi, t) \\ & \quad \quad \quad i \hbar |\psi_t\rangle_t = H |\psi_t\rangle \end{aligned} \quad (60)$$

$$\begin{aligned} > \text{simplify}((59), \{(60)\}, \left\{ \frac{\partial}{\partial t} \text{Ket}(\psi, t) \right\}) \\ & \quad \quad \quad i \hbar |\phi_t\rangle_t = i \hbar U_t |\psi_t\rangle + U H |\psi_t\rangle \end{aligned} \quad (61)$$

Now, from

$$\begin{aligned} > (58) \\ & \quad \quad \quad |\phi_t\rangle = U |\psi_t\rangle \end{aligned} \quad (62)$$

$$\begin{aligned} > U(t)^* \cdot (\text{rhs} = \text{lhs})((58)) \\ & \quad \quad \quad U^\dagger U |\psi_t\rangle = U^\dagger |\phi_t\rangle \end{aligned} \quad (63)$$

$$\begin{aligned} > \text{simplify}((63)) \\ & \quad \quad \quad |\psi_t\rangle = U^\dagger |\phi_t\rangle \end{aligned} \quad (64)$$

Inserting this result in (61)

$$\begin{aligned} > \text{subs}((64), (61)) \\ & \quad \quad \quad i \hbar |\phi_t\rangle_t = i \hbar U_t (U^\dagger |\phi_t\rangle) + U H (U^\dagger |\phi_t\rangle) \end{aligned} \quad (65)$$

the \mathcal{H} amiltonian for $|\phi_t\rangle$ is given by the coefficient of $|\phi_t\rangle$ on the right-hand side

$$\begin{aligned} > \mathcal{H}(t) = \text{Coefficients}(\text{rhs}((65)), \text{Ket}(\phi, t)) \\ & \quad \quad \quad \mathcal{H} = i \hbar U_t U^\dagger + U H U^\dagger \end{aligned} \quad (66)$$

So $|\phi_t\rangle$ satisfies a Schrodinger equation and as one can expect, \mathcal{H} is Hermitian

$$\begin{aligned} > \text{Dagger}((66)) - (66) \\ & \quad \quad \quad \mathcal{H}^\dagger - \mathcal{H} = -i \hbar U U^\dagger_t + U^\dagger U H - i \hbar U_t U^\dagger - U H U^\dagger \end{aligned} \quad (67)$$

Recalling that $U(t)$ satisfies

$$\begin{aligned} > U(t) \cdot U(t)^* = U(t) \cdot U(t)^* \\ & \quad \quad \quad U U^\dagger = 1 \end{aligned} \quad (68)$$

$$> \text{diff}((68), t)$$

$$U_t U^\dagger + U U_t^\dagger = 0 \quad (69)$$

> subs((69), (67))

$$\mathcal{H}^\dagger - \mathcal{H} = -i\hbar U U_t^\dagger + U^\dagger U H - i\hbar U_t U^\dagger - U H U^\dagger \quad (70)$$

In the time independent case, i.e. $U(t) = U$, \mathcal{H} reduced to:

$$\begin{aligned} U &= U \\ \mathcal{H} & \end{aligned} \quad (71)$$

> subs($U(t) = U$, (66))

$$\mathcal{H} = i\hbar U_t U^\dagger + U H U^\dagger \quad (72)$$

> %

$$\mathcal{H} = U H U^\dagger \quad (73)$$

>

▼ Translation operators using Dirac notation

In this section, we focus on the operator $T_a = e^{\frac{-i a P}{\hbar}}$

▼ Settings

> restart, with(Physics) : interface(imaginaryunit = i) :

> Setup(realobjects = {a, x, ħ, m, x₁, x₂}, unitaryoperators = {T}, hermitianoperators = {I, X, P}, quantumcontinuousbasis = {X, P})

[hermitianoperators = {I, P, X}, quantumcontinuousbasis = {P, X}, realobjects = {ħ, a, m, x, x₁, x₂}, unitaryoperators = {T}] (74)

> Setup(bracketrules = { Bracket(Bra(P, p), Ket(ψ)) = ψ̃(p), Bracket(Bra(X, x), Ket(ψ)) = ψ(x), Bracket(Bra(X, x), Ket(P, p)) = (2·π·ħ)^{-1/2} · exp(i/h · x·p) })

[bracketrules = { ⟨P_p | ψ⟩ = ψ̃(p), ⟨X_x | ψ⟩ = ψ(x), ⟨X_x | P_p⟩ = $\frac{\sqrt{2}}{2} \frac{e^{\frac{i x p}{\hbar}}}{\sqrt{\pi \hbar}}$ }] (75)

> Assume(ħ > 0)

$$\{\hbar::(0, \infty)\} \quad (76)$$

Useful closure relations

> 1 = Projector(Ket(X, x))

$$1 = \int_{-\infty}^{\infty} |X_x\rangle \langle X_x| dx \quad (77)$$

To have equivalent projectors with different integration variables, we use \mathbb{I} as the identity operator: $\mathbb{I}^{-1} = \mathbb{I}$, $\mathbb{I}_1 = \mathbb{I}_2$:

> $\mathbb{I}[1] = \text{Projector}(\text{Ket}(P, p))$, $\mathbb{I}[2] = \text{Projector}(\text{Ket}(P, q))$

$$\mathbb{I}_1 = \int_{-\infty}^{\infty} |P_p\rangle \langle P_p| dp, \mathbb{I}_2 = \int_{-\infty}^{\infty} |P_q\rangle \langle P_q| dq \quad (78)$$

>

▼ The Action (translation) of the operator $T_a = e^{-i \frac{aP}{\hbar}}$ on a ket

Considering a general ket $|\psi\rangle$, introduce a closure relation

> $\text{Ket}(\psi) = \mathbb{I}_1 \cdot \text{Ket}(\psi)$

$$|\psi\rangle = \mathbb{I}_1 |\psi\rangle \quad (79)$$

> subs((78), %)

$$|\psi\rangle = \left(\int_{-\infty}^{\infty} |P_p\rangle \langle P_p| dp \right) |\psi\rangle \quad (80)$$

> Bra(X, x) . %

$$\psi(x) = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}} e^{\frac{ixp}{\hbar}} \tilde{\psi}(p)}{2} dp \quad (81)$$

Which gives after a variable change $x = y - a$

> PDEtools:-dchange(x = y - a, %, {y}, known = ψ) : subs(y = x, %)

$$\psi(x - a) = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}} e^{\frac{i(x-a)p}{\hbar}} \tilde{\psi}(p)}{2} dp \quad (82)$$

Let's now evaluate the action of $e^{-i \frac{aP}{\hbar}}$ on $|\psi\rangle$ in the $|X, x\rangle$ basis

> (80)

$$|\psi\rangle = \left(\int_{-\infty}^{\infty} |P_p\rangle \langle P_p| dp \right) |\psi\rangle \quad (83)$$

> Bra(X, x) . $e^{-i \frac{aP}{\hbar}}$. %

$$\left\langle X_x \left| e^{-i \frac{aP}{\hbar}} \right| \psi \right\rangle = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}} e^{\frac{-ip(a-x)}{\hbar}} \tilde{\psi}(p)}{2} dp \quad (84)$$

Comparing the above with (82)

> % - (82)

$$\left\langle X_x \left| e^{\frac{-i a P}{\hbar}} \right| \Psi \right\rangle - \Psi(x - a) = \int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}} e^{\frac{-i p (a - x)}{\hbar}} \tilde{\Psi}(p)}{2} dp - \left(\int_{-\infty}^{\infty} \frac{\sqrt{2} \sqrt{\frac{1}{\pi \hbar}} e^{\frac{i (x - a) p}{\hbar}} \tilde{\Psi}(p)}{2} dp \right) \quad (85)$$

> simplify((85))

$$\left\langle X_x \left| e^{\frac{-i a P}{\hbar}} \right| \Psi \right\rangle - \Psi(x - a) = 0 \quad (86)$$

> isolate(% , $\Psi(x - a)$)

$$\Psi(x - a) = \left\langle X_x \left| e^{\frac{-i a P}{\hbar}} \right| \Psi \right\rangle \quad (87)$$

>

▼ Action of T_a on an operator $V(X)$

Let's consider an operator $V(X)$, that can be written as a formal power series:

$$V(x) = \sum_{n=0}^{\infty} v_n \cdot x^n.$$

Its matrix elements are:

> (%Bracket = Bracket) (Bra(X, x_1), $V(X)$, Ket(X, x_2))

$$\left\langle X_{x_1} \left| V(X) \right| X_{x_2} \right\rangle = V(x_2) \delta(x_2 - x_1) \quad (88)$$

Using the closure relation

> (77)

$$1 = \int_{-\infty}^{\infty} |X_x\rangle \langle X_x| dx \quad (89)$$

$V(X)$ can also be represented in the $|X, x\rangle$ basis as

> $V(X) \cdot$ (77)

$$V(X) = \int_{-\infty}^{\infty} V(x) |X_x\rangle \langle X_x| dx \quad (90)$$

Let's now introduce two closure relations to evaluate $V(X)$ in the momentum basis $|P\rangle$

> [$Ket(X, x) = \mathbb{I}_1 \cdot Ket(X, x)$, $Bra(X, x) = Bra(X, x) \cdot \mathbb{I}_2$]

$$|X_x\rangle = \mathbb{I}_1 |X_x\rangle, \langle X_x| = \langle X_x| \mathbb{I}_2 \quad (91)$$

> subs(% , (90))

$$V(X) = \int_{-\infty}^{\infty} V(x) \mathbb{I}_1 |X_x\rangle (\langle X_x| \mathbb{I}_2) dx \quad (92)$$

Recalling

> (78)

$$I_1 = \int_{-\infty}^{\infty} |P_p\rangle \langle P_p| dp, I_2 = \int_{-\infty}^{\infty} |P_q\rangle \langle P_q| dq \quad (93)$$

> subs(%, %%)

$$V(X) = \int_{-\infty}^{\infty} V(x) \left(\int_{-\infty}^{\infty} |P_p\rangle \langle P_p| dp \right) |X_x\rangle \langle X_x| \left(\int_{-\infty}^{\infty} |P_q\rangle \langle P_q| dq \right) dx \quad (94)$$

> combine((94))

$$V(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(x) |P_p\rangle \langle P_p| |X_x\rangle \langle X_x| |P_q\rangle \langle P_q| dq dp dx \quad (95)$$

> eval(%, '*' = '.')

$$V(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(x) e^{\frac{-ix(p-q)}{\hbar}} |P_p\rangle \langle P_q|}{2\pi\hbar} dq dp dx \quad (96)$$

Apply now the translation operator T_a

$$> T[a] = \exp\left(-\frac{i}{\hbar} \cdot a \cdot P\right)$$

$$T_a = e^{\frac{-iaP}{\hbar}} \quad (97)$$

> %. (96) . %*

$$T_a V(X) T_a^\dagger = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(x) |P_p\rangle \langle P_q| e^{\frac{-i(p-q)(a+x)}{\hbar}}}{2\pi\hbar} dq dp dx \quad (98)$$

Making a variable change $x = y - a$

> PDEtools:-dchange($x = y - a$, %, {y}, known = V) : subs($y = x$, %)

$$T_a V(X) T_a^\dagger = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(x-a) |P_p\rangle \langle P_q| e^{\frac{-ix(p-q)}{\hbar}}}{2\pi\hbar} dq dp dx \quad (99)$$

Evaluate the matrix element of this result and compute the integral

> Bra(X, x_1) • % • Ket(X, x_2)

$$\langle X_{x_1} | T_a V(X) T_a^\dagger | X_{x_2} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{V(x-a) e^{\frac{i(-x+x_1)p + i(x-x_2)q}{\hbar}}}{4\pi^2\hbar^2} dq dp dx \quad (100)$$

dp dx

> value(%)

$$\langle X_{x_1} | T_a V(X) T_a^\dagger | X_{x_2} \rangle = V(x_1 - a) \delta(x_1 - x_2) \quad (101)$$

>

▼ *Quantization of the energy of a particle in a magnetic field

Show that the energy of a particle of charge q and mass m in a constant magnetic field B oriented along the z axis can be written as

$$H = \hbar \omega_c \left(a^\dagger a + \frac{1}{2} \right)$$

where a^\dagger and a are creation and annihilation operators and $\omega_c = \frac{q B}{m}$

>

▼ Solution

The classical Hamiltonian is given by

$$H = \frac{\left(\vec{p} - \frac{q \vec{A}}{c} \right)^2}{2 m}$$

The underlying quantum mechanics algebra rules are

$$[(\vec{r})_i, (\vec{p})_j]_- = \delta_{i,j}, \quad [(\vec{r})_i, (\vec{r})_j]_- = 0, \quad [(\vec{p})_i, (\vec{p})_j]_- = 0$$

> restart, with (Physics) : with (Vectors) : interface(imaginaryunit = i) :

> Setup(hermitianoperators = { $\vec{A}, H, \Pi, \vec{\Pi}, p, \vec{p}, x, y, z$ }, quantumoperators = { a },

realobjects = { $\hbar, B, c, m, q, \omega_c$ });

$$[hermitianoperators = \{\vec{A}, H, \Pi, \vec{\Pi}, p, \vec{p}, x, y, z\}, quantumoperators = \{\vec{A}, H, \Pi, \vec{\Pi}, a, \quad (102)$$

$$p, \vec{p}, x, y, z\}, realobjects = \{\hbar, B, \hat{i}, \hat{j}, \hat{k}, \hat{\phi}, \hat{r}, \hat{\rho}, \hat{\theta}, c, m, \phi, q, r, \rho, \theta, x, y, z, \omega_c\}]$$

Using

$$> \vec{\Pi} = \vec{p} - \frac{q}{c} \cdot \vec{A}_-(x, y)$$

$$\vec{\Pi} = \vec{p} - \frac{q \vec{A}(x, y)}{c} \quad (103)$$

The Hamiltonian can be written as

$$> H = \frac{\vec{\Pi}^2}{2 m}$$

$$H = \frac{\vec{\Pi}^2}{2 m} \quad (104)$$

$$[\Pi_x, \Pi_y]_- = \frac{i q B \hbar}{c} \quad (116)$$

> Setup((116))

$$\left[\text{algebra rules} = \left\{ [x, p_x]_- = i \hbar, [x, p_y]_- = 0, [y, x]_- = 0, [y, p_x]_- = 0, [y, p_y]_- = i \hbar, [\Pi_x, \Pi_y]_- = \frac{i q B \hbar}{c}, [p_y, p_x]_- = 0 \right\} \right. \quad (117)$$

Time to bring in annihilation and creation operators

$$a = \frac{\sqrt{c}}{\sqrt{2 \cdot \hbar \cdot q \cdot B}} (\Pi_x + i \cdot \Pi_y) \quad (118)$$

$$a = \frac{\sqrt{2} \sqrt{c} (\Pi_x + i \Pi_y)}{2 \sqrt{\hbar q B}}$$

> (118)*

$$a^\dagger = \frac{\sqrt{2} \sqrt{c} (\Pi_x - i \Pi_y)}{2 \sqrt{\hbar q B}} \quad (119)$$

Verify the normalization of this definition

> Commutator((118), (119))

$$[a, a^\dagger]_- = 1 \quad (120)$$

> Setup((120))

$$\left[\text{algebra rules} = \left\{ [a, a^\dagger]_- = 1, [x, p_x]_- = i \hbar, [x, p_y]_- = 0, [y, x]_- = 0, [y, p_x]_- = 0, [y, p_y]_- = i \hbar, [\Pi_x, \Pi_y]_- = \frac{i q B \hbar}{c}, [p_y, p_x]_- = 0 \right\} \right. \quad (121)$$

To express the Hamiltonian in terms of a, a^\dagger

> (104)

$$H = \frac{\overset{\rightarrow}{\Pi}^2}{2 m} \quad (122)$$

> subs((110), %)

$$H = \frac{(\hat{i} \Pi_x + \hat{j} \Pi_y)^2}{2 m} \quad (123)$$

> {(118), (119)}

$$\left\{ a = \frac{\sqrt{2} \sqrt{c} (\Pi_x + i \Pi_y)}{2 \sqrt{\hbar q B}}, a^\dagger = \frac{\sqrt{2} \sqrt{c} (\Pi_x - i \Pi_y)}{2 \sqrt{\hbar q B}} \right\} \quad (124)$$

> solve(%, {\Pi_x, \Pi_y})

(125)

$$\left\{ \Pi_x = \frac{\sqrt{\hbar q B} (a^\dagger + a) \sqrt{2}}{2\sqrt{c}}, \Pi_y = \frac{\frac{i}{2} \sqrt{\hbar q B} (a^\dagger - a) \sqrt{2}}{\sqrt{c}} \right\} \quad (125)$$

> subs((125), (123))

$$H = \frac{\left(\frac{\hat{i} \sqrt{\hbar q B} (a^\dagger + a) \sqrt{2}}{2\sqrt{c}} + \frac{\hat{j} \sqrt{\hbar q B} (a^\dagger - a) \sqrt{2}}{2\sqrt{c}} \right)^2}{2m} \quad (126)$$

> simplify(expand((126)))

$$H = \frac{\hbar q B (-1 + 2 a a^\dagger)}{2 m c} \quad (127)$$

> Library:-SortProducts((127), [Dagger(a), a], usecommutator)

$$H = \frac{\hbar q B (1 + 2 a^\dagger a)}{2 m c} \quad (128)$$

This is the Hamiltonian of an harmonic oscillator with frequency $\omega_c = \frac{q B}{m}$. The possible values for the energy are known: $E = \hbar \omega_c \left(n + \frac{1}{2} \right)$, where n is a positive integer.

>

▼ Classical Field Theory

▼ The field equations for the $\lambda \Phi^4$ model

The Lagrangian of the $\lambda \Phi^4$ model, the corresponding Action, and the field equations:

> restart, with(Physics) :

> Coordinates(cartesian)

Default differentiation variables for d , D and $dAlembertian$ are: $\{X = (x, y, z, t)\}$

Systems of spacetime Coordinates are: $\{X = (x, y, z, t)\}$

$$\{X\} \quad (129)$$

> CompactDisplay($\Phi(X)$)

$$\Phi(x, y, z, t) \text{ will now be displayed as } \Phi \quad (130)$$

$$> L := \frac{1}{2} d_{-\mu}(\Phi(X))^2 - \frac{m^2}{2} \Phi(X)^2 + \left(\frac{\lambda}{4} \Phi(X)^4 \right)$$

$$L := \frac{\partial_{-\mu}(\Phi) \partial^{\mu}(\Phi)}{2} - \frac{m^2 \Phi^2}{2} + \frac{\lambda \Phi^4}{4} \quad (131)$$

> $S := \text{Intc}(L, X)$

$$S := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\partial_{-\mu}(\Phi) \partial^{\mu}(\Phi)}{2} - \frac{m^2 \Phi^2}{2} + \frac{\lambda \Phi^4}{4} \right) dx dy dz dt \quad (132)$$

> (%Fundiff= Fundiff) (S, Φ)

$$\left(\frac{\delta}{\delta \Phi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\partial_{\mu}(\Phi) \partial^{\mu}(\Phi)}{2} - \frac{m^2 \Phi^2}{2} + \frac{\lambda \Phi^4}{4} \right) dx dy dz dt = -\square(\Phi) \quad (133)$$

$$- \Phi (-\Phi^2 \lambda + m^2)$$

> convert(%, diff)

$$\left(\frac{\delta}{\delta \Phi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\Phi_{(X) \sim \mu} \Phi_{(X) \mu}}{2} - \frac{m^2 \Phi^2}{2} + \frac{\lambda \Phi^4}{4} \right) dx dy dz dt = \Phi_{x,x} \quad (134)$$

$$+ \Phi_{y,y} + \Phi_{z,z} - \Phi_{t,t} - \Phi (-\Phi^2 \lambda + m^2)$$

▼ *Maxwell equations departing from the 4-dimensional Action for Electrodynamics

Maxwell equations result from equating to zero the functional derivative of the Action with respect to the 4-D potential A_{μ}

> restart, with(Physics) :

> Coordinates(X= Cartesian)

Default differentiation variables for d_, D_ and dAlembertian are: {X= (x, y, z, t)}

Systems of spacetime Coordinates are: {X= (x, y, z, t)}

$$\{X\} \quad (135)$$

The 4D electromagnetic potential

> Define(A[mu](X))

Defined objects with tensor properties

$$\{A_{\mu}, \gamma_{\mu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, g_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\} \quad (136)$$

> CompactDisplay(A(X))

$$A(x, y, z, t) \text{ will now be displayed as } A \quad (137)$$

The electromagnetic field tensor $F_{\mu, \nu}$

> F[mu, nu] := d_[mu](A[nu](X)) - d_[nu](A[mu](X));

$$F_{\mu, \nu} := \partial_{\mu}(A_{\nu}) - \partial_{\nu}(A_{\mu}) \quad (138)$$

The functional derivative of the corresponding Action

> 'Fundiff'(Intc(F[mu, nu]^2, X), A[rho]) = 0

$$\left(\frac{\delta}{\delta A_{\rho}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\partial_{\mu}(A_{\nu}) - \partial_{\nu}(A_{\mu}))^2 dx dy dz dt = 0 \quad (139)$$

> (139)

$$(140)$$

$$\left(2 \partial_{\mu} \left(\partial_{\nu} \left(A^{\nu}\right)\right) - 2 \square \left(A_{\mu}\right)\right) g^{\mu, \rho} + \left(-2 \square \left(A_{\nu}\right) + 2 \partial_{\mu} \left(\partial_{\nu} \left(A^{\mu}\right)\right)\right) g^{\nu, \rho} = 0 \quad (140)$$

Simplify the contracted spacetime indices

> *Simplify*((140))

$$-4 \square \left(A^{\rho}\right) + 4 \partial^{\mu} \left(\partial^{\rho} \left(A_{\mu}\right)\right) = 0 \quad (141)$$

The system of differential equations behind this formula in standard Maple notation:

> *Library:-ToContravariant*((141))

$$4 g_{\mu, \nu} \partial^{\mu} \left(\partial^{\rho} \left(A^{\nu}\right)\right) - 4 \square \left(A^{\rho}\right) = 0 \quad (142)$$

> *convert*(*Library:-TensorComponents* ((142)), *diff*)

$$\begin{aligned} & \left[-4 \left(A^2\right)_{x, y} - 4 \left(A^3\right)_{x, z} - 4 \left(A^4\right)_{t, x} + 4 \left(A^1\right)_{y, y} + 4 \left(A^1\right)_{z, z} - 4 \left(A^1\right)_{t, t} = 0, \right. \\ & -4 \left(A^1\right)_{x, y} - 4 \left(A^3\right)_{y, z} - 4 \left(A^4\right)_{t, y} + 4 \left(A^2\right)_{x, x} + 4 \left(A^2\right)_{z, z} - 4 \left(A^2\right)_{t, t} = 0, \\ & -4 \left(A^1\right)_{x, z} - 4 \left(A^2\right)_{y, z} - 4 \left(A^4\right)_{t, z} + 4 \left(A^3\right)_{x, x} + 4 \left(A^3\right)_{y, y} - 4 \left(A^3\right)_{t, t} = 0, \\ & \left. 4 \left(A^1\right)_{t, x} + 4 \left(A^2\right)_{t, y} + 4 \left(A^3\right)_{t, z} + 4 \left(A^4\right)_{x, x} + 4 \left(A^4\right)_{y, y} + 4 \left(A^4\right)_{z, z} = 0 \right], \end{aligned} \quad (143)$$

>

▼ *The Gross-Pitaevskii field equations for a quantum system of identical particles

Problem: derive the field equation describing the ground state of a quantum system of identical particles (bosons), that is, the Gross-Pitaevskii equation (GPE). This equation is useful to describe Bose-Einstein condensates (BEC).

▼ *Solution*

Two steps:

- Construct the Lagrangian for the system, and with it write the action functional
- Minimize the action by equating to zero its functional derivative with respect to the boson field.

> *restart, with* (*Physics*) : *with* (*Physics* [*Vectors*]) :

> *interface* (*imaginaryunit* = *i*) :

> *macro* (*Psi* = *psi* (*x, y, z, t*)) :

> *CompactDisplay* (*(psi, V)* (*x, y, z, t*))

psi (*x, y, z, t*) will now be displayed as ψ

V (*x, y, z, t*) will now be displayed as V

(144)

The energy density E for a quantum system of identical boson particles is (see [3])

$$> E := \frac{\hbar^2}{2m} \text{Norm}(\% \text{Gradient}(\text{Psi}))^2 + V(x, y, z, t) \text{abs}(\text{Psi})^2 + \frac{G}{2} \text{abs}(\text{Psi})^4;$$

$$E := \frac{\hbar^2 \|\nabla \Psi\|^2}{2m} + V|\Psi|^2 + \frac{G|\Psi|^4}{2} \quad (145)$$

$\Psi(x, y, z, t)$ is a complex field, $V(x, y, z, t)$ an external potential, the term $\frac{G|\Psi|^4}{2}$ is the atom-atom interaction.

> *Setup*(*realobjects* = { $t, m, \hbar, G, V(x, y, z, t)$ }, *automaticsimplication* = true) :

The Lagrangian density L in terms of the Energy E according to standard formulas

$$> L := \left(\frac{i\hbar}{2} \right) (\text{conjugate}(\text{Psi}) \text{diff}(\text{Psi}, t) - \text{Psi} \text{diff}(\text{conjugate}(\text{Psi}), t)) - E$$

$$L := \frac{-i\hbar\Psi\bar{\Psi}_t m - \hbar^2 \|\nabla\Psi\|^2 + (-G|\Psi|^4 + i\Psi_t\bar{\Psi}\hbar - 2V|\Psi|^2)m}{2m} \quad (146)$$

Construct the action and equate to zero the functional derivative

> 'Fundiff'(Intc(L, x, y, z, t), psi) = 0

$$\left(\frac{\delta}{\delta\Psi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \quad (147)$$

$$\int_{-\infty}^{\infty} \frac{-i\hbar\Psi\bar{\Psi}_t m - \hbar^2 \|\nabla\Psi\|^2 + (-G|\Psi|^4 + i\Psi_t\bar{\Psi}\hbar - 2V|\Psi|^2)m}{2m} dx dy dz dt = 0$$

> (147)

$$\frac{-2G\Psi\bar{\Psi}^2 m - 2i\hbar\bar{\Psi}_t m + \hbar^2\bar{\Psi}_{x,x} + \bar{\Psi}_{y,y}\hbar^2 + \hbar^2\bar{\Psi}_{z,z} - 2V\bar{\Psi}m}{2m} = 0 \quad (148)$$

Make the Laplacian explicit

> (Laplacian = %Laplacian)(Psi)

$$\Psi_{x,x} + \Psi_{y,y} + \Psi_{z,z} = \nabla^2\Psi \quad (149)$$

> simplify(conjugate((148)), {(149)})

$$\frac{2i\hbar\Psi_t m + \hbar^2\nabla^2\Psi - 2m\Psi(G\bar{\Psi}\Psi + V)}{2m} = 0 \quad (150)$$

The standard form of the Gross–Pitaevskii equation:

> $i\hbar$ isolate((150), diff(Psi, t))

$$i\hbar\Psi_t = \frac{-\hbar^2\nabla^2\Psi + 2m\Psi(G\bar{\Psi}\Psi + V)}{2m} \quad (151)$$

> collect(convert(expand((151)), abs), psi)

$$i \hbar \psi_t = (G |\psi|^2 + V) \psi - \frac{\hbar^2 \nabla^2 \psi}{2m} \quad (152)$$

- For a continuation of this computation deriving a continuity equation for a system of identical particles, see the Mapleprimes post "[Quantum Mechanics using Computer Algebra](#)".
- For the Bogoliubov spectrum and dispersion relations of this problem above see the Mapleprimes post "[Quantum Mechanics II](#)".
- For a derivation of the Landau criterion for superfluidity in a system of identical particles see the Mapleprimes post "[Superfluidity in Quantum Mechanics](#)".

>

▼ General Relativity

▼ Exact Solutions to Einstein's Equations $g_{\mu, \nu} \Lambda + G_{\mu, \nu} = 8 \pi T_{\mu, \nu}$

Main reference: - Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. Herlt, E. **Exact Solutions of Einstein's Field Equations**, Cambridge Monographs on Mathematical Physics, second edition. Cambridge University Press, 2003.

The authors reviewed more than 4,000 papers containing solutions to Einstein's equations in the literature and organized the material into chapters according to the physical properties of these solutions.

These solutions are digitized within Maple since 2016, so that ***it is now possible to actually compute with them.***

▼ Examples

Load [Physics](#), set the metric to be *Schwarzschild's* solution (and everything else automatically) in one go

> restart, with(Physics) :

> g_[sc]

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for $d_$, $D_$ and d Alembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (153)$$

And that is all we do.

The tensor components of the general relativity tensors related to this solution get derived automatically from their definition

> *Christoffel[definition]*

$$\Gamma_{\alpha, \mu, \nu} = \frac{\partial_\nu (g_{\alpha, \mu})}{2} + \frac{\partial_\mu (g_{\alpha, \nu})}{2} - \frac{\partial_\alpha (g_{\mu, \nu})}{2} \quad (154)$$

> *Riemann[definition]*

$$R_{\alpha, \beta, \mu, \nu} = g_{\alpha, \lambda} \left(\partial_\mu (\Gamma_{\beta, \nu}^\lambda) - \partial_\nu (\Gamma_{\beta, \mu}^\lambda) + \Gamma_{\nu, \mu}^\lambda \Gamma_{\beta, \nu}^\nu - \Gamma_{\nu, \nu}^\lambda \Gamma_{\beta, \mu}^\nu \right) \quad (155)$$

For example, the Riemann invariants using the standard formulas by [Carminati and McLenaghan](#)

> *Riemann[invariants]*

$$r_0 = 0, r_1 = 0, r_2 = 0, r_3 = 0, w_1 = \frac{6m^2}{r^6}, w_2 = \frac{6m^3}{r^9}, m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 0, \quad (156)$$

$$m_5 = 0$$

The related [Weyl scalars](#) in the context of the [Newman-Penrose formalism](#)

> *Weyl[scalarsdefinition]*

$$\Psi_0 = -C^{\mu, \nu, \alpha, \beta} l_\mu m_\nu l_\alpha m_\beta, \Psi_1 = -C^{\mu, \nu, \alpha, \beta} l_\mu n_\nu l_\alpha m_\beta, \Psi_2 = -C^{\mu, \nu, \alpha, \beta} l_\mu m_\nu \bar{m}_\alpha n_\beta, \Psi_3 = -C^{\mu, \nu, \alpha, \beta} l_\mu n_\nu \bar{m}_\alpha n_\beta, \Psi_4 = -C^{\mu, \nu, \alpha, \beta} n_\mu \bar{m}_\nu n_\alpha \bar{m}_\beta \quad (157)$$

> *Weyl[scalars]*

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = -\frac{m}{r^3}, \Psi_3 = 0, \Psi_4 = 0 \quad (158)$$

The Killing vectors

> *Define(K, quiet)* :

> *KillingVectors(K)*

$$\left[K^\mu = [0, 0, 0, 1], K^\mu = \left[0, \sin(\phi), \frac{\cos(\phi)}{\tan(\theta)}, 0 \right], K^\mu = \left[0, \cos(\phi), -\frac{\sin(\phi)}{\tan(\theta)}, 0 \right], K^\mu = [0, 0, 1, 0] \right] \quad (159)$$

These are the 2x2 matrix components of the [Christoffel symbols of the second kind](#) with the

first index contravariant with value 1

> *Christoffel*[~1, alpha, beta, matrix]

$$\Gamma^1_{\alpha, \beta} = \begin{pmatrix} \frac{m}{r(-r+2m)} & 0 & 0 & 0 \\ 0 & -r+2m & 0 & 0 \\ 0 & 0 & (-r+2m)\sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{-2m^2+rm}{r^3} \end{pmatrix} \quad (160)$$

One can query the database, directly from the spacetime metrics command (*g_*).

For example, these are the solutions (metrics) to Einstein's equations that appear in the book and related to Levi-Civita, the Italian mathematician

> *g_*[civi]

```
[12, 16, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)",
"Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell",
"SecondaryDescription" = ["Homogeneous" ]]
```

```
[12, 18, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)",
"Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell",
"SecondaryDescription" = ["Homogeneous" ]]
```

```
[12, 19, 1] = ["Authors" = ["Bertotti (1959)", "Kramer (1978)", "Levi-Civita (1917)",
"Robinson (1959)"], "PrimaryDescription" = "EinsteinMaxwell",
"SecondaryDescription" = ["Homogeneous"], "Comments" = ["_lambda=_zeta" ]]
```

```
[22, 7, 1] = ["Authors" = ["Levi-Civita (1917), Frehland (1971)"],
"PrimaryDescription" = "Vacuum", "SecondaryDescription"
= ["Cylindrically-Symmetric"], "Comments"
= ["Locally static, Weyl class_m=0,1 - flat, _m=1/2, 2, -1 - PetrovType D" ]]
```

Warning, found more than one match for the keyword 'civi', as seen above. Please refine your 'keyword' or re-enter the metric '*g_*[...]' with the list of three numbers identifying the metric, for example as in *g_*[12, 16, 1] or *Setup*(metric = [12, 16, 1]) (161)

These solutions can be set in one go from the metrics command, just by indicating the number with which it appears in the "Exact Solutions to Einstein's Equations" book.

For example, set one of these solutions and everything related in one go

> `g_([28, 74, 1])`

Systems of spacetime Coordinates are: $\{X = (u, \eta, r, y)\}$

Default differentiation variables for `d_`, `D_` and `dAlembertian` are: $\{X = (u, \eta, r, y)\}$

The Frolov and Khlebnikov (1975) metric in coordinates $[u, \eta, r, y]$

Parameters: $[\kappa_0, m(u), b, d]$

Comments: With $m(u) = \text{constant}$, the metric is Ricci flat and becomes 28.24 in Stephani.

Resetting the signature of spacetime from "- - - +" to "- + + +" in order to match the signature in the database of metrics:

$$g_{\mu, \nu} = \begin{bmatrix} \left[\frac{2 m(u)^3 - 6 m(u)^2 \eta r - r^2 (-6 \eta^2 + b) m(u) + r^3 (-2 \eta^3 + b \eta + d)}{r m(u)^2}, \right. \\ \left. -\frac{r^2}{m(u)}, -1, 0 \right], \\ \left[-\frac{r^2}{m(u)}, \frac{r^2}{-2 \eta^3 + b \eta + d}, 0, 0 \right], \\ \left[-1, 0, 0, 0 \right], \\ \left[0, 0, 0, r^2 (-2 \eta^3 + b \eta + d) \right] \end{bmatrix} \quad (162)$$

The amount of solutions/cases found in the book and digitized in Maple during 2015

> `nops(DifferentialGeometry:-Library:-Retrieve("Stephani", 1))`

991

(163)

Computational challenge: load again Schwarzschild's solution, rewrite Einstein's equations in terms of the metric $g_{\mu, \nu}$ and show that all the components of the *EnergyMomentum*

tensor $T_{\mu, \nu}$ are equal to zero (Schwarzschild's solution in vacuum)

> `g_[sc]`

Systems of spacetime Coordinates are: $\{X = (t, r, \theta, \phi)\}$

Default differentiation variables for `d_`, `D_` and `dAlembertian` are: $\{X = (t, r, \theta, \phi)\}$

The Schwarzschild metric in coordinates $[t, r, \theta, \phi]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{-r + 2m}{r} & 0 & 0 & 0 \\ 0 & -\frac{r}{-r + 2m} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix} \quad (164)$$

> *EnergyMomentum* [definition]

$$T_{\mu, \nu} = \frac{G_{\mu, \nu}}{8 \pi} \quad (165)$$

where

> *Einstein* [definition]

$$G_{\mu, \nu} = R_{\mu, \nu} - \frac{g_{\mu, \nu} R_{\alpha}^{\alpha}}{2} \quad (166)$$

Rewrite the right-hand side in terms of the metric $g_{\mu, \nu}$

> *lhs* (%) = *convert*(*rhs* (%), *g_*, *evaluatetrace*)

$$G_{\mu, \nu} = \frac{\partial_{\beta}(g^{\beta, \lambda}) (\partial_{\nu}(g_{\lambda, \mu}) + \partial_{\mu}(g_{\lambda, \nu}) - \partial_{\lambda}(g_{\mu, \nu}))}{2} \quad (167)$$

$$+ \frac{g^{\beta, \lambda} (\partial_{\beta}(\partial_{\nu}(g_{\lambda, \mu})) + \partial_{\beta}(\partial_{\mu}(g_{\lambda, \nu})) - \partial_{\beta}(\partial_{\lambda}(g_{\mu, \nu})))}{2}$$

$$- \frac{\partial_{\nu}(g^{\beta, \lambda}) \partial_{\mu}(g_{\beta, \lambda})}{2} - \frac{g^{\beta, \lambda} \partial_{\mu}(\partial_{\nu}(g_{\beta, \lambda}))}{2}$$

$$+ \frac{g^{\kappa, \sigma} (\partial_{\nu}(g_{\mu, \sigma}) + \partial_{\mu}(g_{\nu, \sigma}) - \partial_{\sigma}(g_{\mu, \nu})) g^{\beta, \lambda} \partial_{\kappa}(g_{\beta, \lambda})}{4} - \frac{1}{4} \left(\right.$$

$$g^{\kappa, \sigma} (\partial_{\mu}(g_{\beta, \sigma}) + \partial_{\beta}(g_{\mu, \sigma}) - \partial_{\sigma}(g_{\beta, \mu})) g^{\beta, \lambda} (\partial_{\nu}(g_{\kappa, \lambda}) + \partial_{\kappa}(g_{\lambda, \nu})$$

$$\left. - \partial_{\lambda}(g_{\kappa, \nu})) \right)$$

> *TensorArray*(*rhs* ((167)), *simplifier* = *simplify*)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (168)$$

>

▼ **"Physical Review D" 87, 044053 (2013)**

Given the spacetime metric,

$$g_{\mu, \nu} = \begin{bmatrix} -e^{\lambda(r)} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & e^{\nu(r)} \end{bmatrix}$$

a) Compute the Ricci and Weyl scalars

b) Compute the trace of

$$Z_{\alpha}^{\beta} = \Phi R_{\alpha}^{\beta} + \mathcal{D}_{\alpha} \mathcal{D}^{\beta} \Phi + T_{\alpha}^{\beta}$$

where $\Phi \equiv \Phi(r)$ is some function of the radial coordinate, R_{α}^{β} is the Ricci tensor, \mathcal{D}_{α} is the covariant derivative operator and T_{α}^{β} is the stress-energy tensor

$$T_{\alpha, \beta} = \begin{bmatrix} 8 e^{\lambda(r)} \pi & 0 & 0 & 0 \\ 0 & 8 r^2 \pi & 0 & 0 \\ 0 & 0 & 8 r^2 \sin(\theta)^2 \pi & 0 \\ 0 & 0 & 0 & 8 e^{\nu(r)} \pi \epsilon \end{bmatrix}$$

c) Compute the components of $W_{\alpha}^{\beta} \equiv$ the traceless part of Z_{α}^{β} of item b)

d) Compute an exact solution to the nonlinear system of differential equations conformed by the components of W_{α}^{β} obtained in c)

Background: paper from February/2013, "[Withholding Potentials, Absence of Ghosts and Relationship between Minimal Dilatonic Gravity and f\(R\) Theories](#)", by P. Fiziev.

▼ **a) The Ricci and Weyl scalars**

> *restart, with (Physics) :*

Set the coordinates

> *Setup(coordinates = spherical, automaticsimplication = true)*

** Partial match of 'coordinates' against keyword 'coordinatesystems'*

Default differentiation variables for d_, D_ and dAlembertian are: {X = (r, θ, φ, t)}

Systems of spacetime Coordinates are: {X = (r, θ, φ, t)}

$$[\text{automaticsimplication} = \text{true}, \text{coordinatesystems} = \{X\}] \quad (169)$$

The square of the line element and the metric

$$\begin{aligned} > ds^2 := \exp(\text{nu}(r))dt^2 - \exp(\text{lambda}(r))dr^2 - r^2 d\theta^2 - r^2 \sin(\theta)^2 d\phi^2 \\ ds^2 &:= -e^{\lambda(r)} dr^2 + e^{\nu(r)} dt^2 - r^2 (d\phi^2 \sin(\theta)^2 + d\theta^2) \end{aligned} \quad (170)$$

$$\begin{aligned} > \text{CompactDisplay}(ds^2) \\ \lambda(r) &\text{ will now be displayed as } \lambda \\ \nu(r) &\text{ will now be displayed as } \nu \end{aligned} \quad (171)$$

$$\begin{aligned} > \text{Setup}(\text{metric} = ds^2) : g_{\mu, \nu} \\ g_{\mu, \nu} &= \begin{bmatrix} -e^\lambda & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & e^\nu \end{bmatrix} \end{aligned} \quad (172)$$

$$\begin{aligned} > \text{with}(\text{Tetrads}) \\ \text{Setting lowercaselatin_ah letters to represent tetrad indices} \\ \text{Defined as tetrad tensors (see ?Physics,tetrads), } e_{a, \mu}, \eta_{a, b}, \gamma_{a, b, c}, \lambda_{a, b, c} \\ \text{Defined as spacetime tensors representing the NP null vectors of the tetrad formalism} \\ \text{(see ?Physics,tetrads), } l_\mu, n_\mu, m_\mu, \bar{m}_\mu \\ [\text{IsTetrad, NullTetrad, OrthonormalTetrad, PetrovType, SegreType, TransformTetrad}, \end{aligned} \quad (173)$$

$$\begin{aligned} > \text{PetrovType}() \\ \text{"D"} \end{aligned} \quad (174)$$

$$\begin{aligned} > e_{\text{nullvectors}} \\ l_\mu &= \begin{bmatrix} \frac{\sqrt{2} e^{\frac{\lambda}{2}}}{2} & 0 & 0 & \frac{\sqrt{2} e^{\frac{\nu}{2}}}{2} \end{bmatrix}, n_\mu = \begin{bmatrix} -\frac{\sqrt{2} e^{\frac{\lambda}{2}}}{2} & 0 & 0 & \frac{\sqrt{2} e^{\frac{\nu}{2}}}{2} \end{bmatrix}, m_\mu \\ &= \begin{bmatrix} 0 & \frac{\sqrt{2} r}{2} & \frac{1}{2} \sqrt{2} r \sin(\theta) & 0 \end{bmatrix}, \bar{m}_\mu = \begin{bmatrix} 0 & \frac{\sqrt{2} r}{2} & -\frac{1}{2} \sqrt{2} r \sin(\theta) & 0 \end{bmatrix} \end{aligned} \quad (175)$$

$$\begin{aligned} > \text{Ricci}[\text{scalarsdefinition}] \\ \Phi_{00} &= -\frac{R^{\mu, \nu} l_\mu l_\nu}{2}, \Phi_{01} = -\frac{R^{\mu, \nu} l_\mu m_\nu}{2}, \Phi_{02} = -\frac{R^{\mu, \nu} m_\mu m_\nu}{2}, \Phi_{11} = \\ &-\frac{R^{\mu, \nu} (l_\mu n_\nu + m_\mu \bar{m}_\nu)}{4}, \Phi_{12} = -\frac{R^{\mu, \nu} m_\mu n_\nu}{2}, \Phi_{22} = -\frac{R^{\mu, \nu} n_\mu n_\nu}{2}, \Lambda = \frac{R^\mu{}_\mu}{24} \end{aligned} \quad (176)$$

$$> \text{Ricci}[\text{scalars}]$$

$$\begin{aligned}
\Phi_{00} &= -\frac{e^{-\lambda} (\lambda_r + v_r)}{4r}, \Phi_{01} = 0, \Phi_{02} = 0, \Phi_{11} \\
&= \frac{-4 + (-v_r^2 r^2 + v_r \lambda_r r^2 - 2 v_{r,r} r^2 + 4) e^{-\lambda}}{16 r^2}, \Phi_{12} = 0, \Phi_{22} = \\
&-\frac{e^{-\lambda} (\lambda_r + v_r)}{4r}, \Lambda \\
&= \frac{e^{-\lambda} (2 v_{r,r} r^2 + v_r^2 r^2 + (-r^2 \lambda_r + 4r) v_r - 4 \lambda_r r - 4 e^\lambda + 4)}{48 r^2}
\end{aligned} \tag{177}$$

> Weyl[scalarsdefinition]

$$\begin{aligned}
\Psi_0 &= -C^{\mu, \nu, \alpha, \beta} l_\mu m_\nu l_\alpha m_\beta, \Psi_1 = -C^{\mu, \nu, \alpha, \beta} l_\mu n_\nu l_\alpha m_\beta, \Psi_2 = - \\
&C^{\mu, \nu, \alpha, \beta} l_\mu m_\nu \bar{m}_\alpha n_\beta, \Psi_3 = -C^{\mu, \nu, \alpha, \beta} l_\mu n_\nu \bar{m}_\alpha n_\beta, \Psi_4 = -C^{\mu, \nu, \alpha, \beta} n_\mu \bar{m}_\nu n_\alpha \bar{m}_\beta
\end{aligned} \tag{178}$$

> Weyl[scalars]

$$\begin{aligned}
\Psi_0 &= 0, \Psi_1 = 0, \Psi_2 \\
&= \frac{e^{-\lambda} (2 v_{r,r} r^2 + v_r^2 r^2 + (-r^2 \lambda_r - 2r) v_r + 2 \lambda_r r - 4 e^\lambda + 4)}{24 r^2}, \Psi_3 = 0, \\
\Psi_4 &= 0
\end{aligned} \tag{179}$$

>

▼ **b) The trace of** $Z_\alpha^\beta = \Phi R_\alpha^\beta + \mathcal{D}_\alpha \mathcal{D}^\beta \Phi + T_\alpha^\beta$

The indicated stress-energy tensor

> $T[\alpha, \beta] = 8 \cdot \text{Pi} \cdot \text{Matrix}(4, \langle \exp(\lambda(r)), r^2, r^2 \sin(\theta)^2, \epsilon \exp(\nu(r)) \rangle, \text{shape} = \text{diagonal})$

$$T_{\alpha, \beta} = \begin{bmatrix} 8 \pi e^\lambda & 0 & 0 & 0 \\ 0 & 8 \pi r^2 & 0 & 0 \\ 0 & 0 & 8 \pi r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & 8 \pi e^\nu \end{bmatrix} \tag{180}$$

> Define((180))

Defined objects with tensor properties

$$\left\{ \mathcal{D}_\mu, \gamma_\mu, \sigma_\mu, R_{\mu, \nu}, R_{\mu, \nu, \alpha, \beta}, T_{\alpha, \beta}, C_{\mu, \nu, \alpha, \beta}, X_\mu, \partial_\mu, e_{a, \mu}, \eta_{a, b}, g_{\mu, \nu}, \gamma_{a, b, c}, l_\mu, \lambda_{a, b, c}, m_\mu, \bar{m}_\mu, n_\mu, \Gamma_{\mu, \nu, \alpha}, G_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu} \right\} \tag{181}$$

Solve item **b)** in one go, that is the trace of Z , by defining the tensorial equation

$Z_{\alpha}^{\beta} = \Phi R_{\alpha}^{\beta} + \mathcal{D}_{\alpha} \mathcal{D}^{\beta} \Phi + T_{\alpha}^{\beta}$ then taking its trace

> *CompactDisplay*(Phi(r))

$$\Phi(r) \text{ will now be displayed as } \Phi \quad (182)$$

> $Z[\mu, \nu] = \Phi(r) \text{ Ricci}[\mu, \nu] + 'D_{-}[\mu](D_{-}[\nu](\Phi(r)))' + T[\mu, \nu]$

$$Z_{\mu, \nu} = \Phi R_{\mu, \nu} + \mathcal{D}_{\mu} \left(\mathcal{D}_{\nu} (\Phi) \right) + T_{\mu, \nu} \quad (183)$$

> *Define*((183))

Defined objects with tensor properties

$$\left\{ \mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu, \nu}, R_{\mu, \nu, \alpha, \beta}, T_{\alpha, \beta}, C_{\mu, \nu, \alpha, \beta}, X_{\mu}, Z_{\mu, \nu}, \partial_{\mu}, e_{a, \mu}, \eta_{a, b}, g_{\mu, \nu}, \gamma_{a, b, c}, l_{\mu}, \lambda_{a, b, c}, m_{\mu}, \bar{m}_{\mu}, n_{\mu}, \Gamma_{\mu, \nu, \alpha}, G_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu} \right\} \quad (184)$$

The answer to a), that is the trace of $Z_{\mu, \nu}$

> $Z[\mu, \mu]$

$$Z_{\mu}^{\mu} \quad (185)$$

> *SumOverRepeatedIndices*((185))

$$\begin{aligned} & \frac{1}{4 r^2} \left(-e^{-\nu} r \left(-2 \Phi v_{r, r} r + v_r \left(\Phi \lambda_r r - \Phi v_r r + 2 r \Phi_r - 4 \Phi \right) \right) e^{-\lambda + \nu} \right. \\ & \quad + \left(2 r^2 v_{r, r} \Phi - 4 r^2 \Phi_{r, r} + r^2 \Phi v_r^2 - r \Phi \left(\lambda_r r - 4 \right) v_r + \left(2 r^2 \Phi_r \right. \right. \\ & \quad \left. \left. - 8 \Phi r \right) \lambda_r - 32 \pi r^2 e^{\lambda} - 8 r \Phi_r + 8 \Phi \right) e^{-\lambda} + 32 e^{\nu} e^{-\nu} \pi \epsilon r^2 - 64 \pi r^2 \\ & \quad \left. - 8 \Phi \right) \end{aligned} \quad (186)$$

>

▼ c) *The components of $W_{\alpha}^{\beta} \equiv$ the traceless part of Z_{α}^{β}*

Define a tensor $W_{\mu, \nu}$ with the traceless part of $Z_{\mu, \nu}$

> $W[\mu, \nu] = Z[\mu, \nu] - \frac{Z[\alpha, \alpha]}{4} g_{-}[\mu, \nu]$

$$W_{\mu, \nu} = Z_{\mu, \nu} - \frac{Z_{\alpha}^{\alpha} g_{\mu, \nu}}{4} \quad (187)$$

> *Define*((187))

Defined objects with tensor properties

$$\left\{ \mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu, \nu}, R_{\mu, \nu, \alpha, \beta}, T_{\alpha, \beta}, W_{\mu, \nu}, C_{\mu, \nu, \alpha, \beta}, X_{\mu}, Z_{\mu, \nu}, \partial_{\mu}, e_{a, \mu}, \eta_{a, b}, g_{\mu, \nu}, \gamma_{a, b, c}, l_{\mu}, \lambda_{a, b, c}, m_{\mu}, \bar{m}_{\mu}, n_{\mu}, \Gamma_{\mu, \nu, \alpha}, G_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu} \right\} \quad (188)$$

Verify that W is traceless

> $W[\mu, \mu]$

(189)

$$W_{\mu}^{\mu} \quad (189)$$

> *SumOverRepeatedIndices* ((189), *simplifier = simplify*)

$$0 \quad (190)$$

The nonzero components for the traceless W_{μ}^{ν} are then

> $W[\text{mu}, \sim \text{nu}, \text{nonzero}]$

$$W_{\mu}^{\nu} = \left\{ (1, 1) = \frac{1}{8 r^2} \left(\left(-6 r^2 \Phi_{r,r} + 2 r^2 v_{r,r} \Phi + r^2 \Phi v_r^2 - r \left(\Phi \lambda_r r - r \Phi_r \right. \right. \right. \right. \quad (191)$$

$$\left. \left. \left. + 4 \Phi \right) v_r + \left(3 r^2 \Phi_r - 4 \Phi r \right) \lambda_r + 4 r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (-16 \epsilon$$

$$- 16) \pi r^2 \right), (2, 2) = \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} - 2 r^2 v_{r,r} \Phi + \left(r^2 v_r - r^2 \lambda_r \right. \right. \right. \right.$$

$$\left. \left. \left. - 4 r \right) \Phi_r - \Phi \left(v_r^2 r^2 - v_r \lambda_r r^2 - 4 \right) \right) e^{-\lambda} - 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right), (3,$$

$$3) = \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} - 2 r^2 v_{r,r} \Phi + \left(r^2 v_r - r^2 \lambda_r - 4 r \right) \Phi_r \right. \right. \right.$$

$$\left. \left. \left. - \Phi \left(v_r^2 r^2 - v_r \lambda_r r^2 - 4 \right) \right) e^{-\lambda} - 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right), (4, 4)$$

$$= \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} + 2 r^2 v_{r,r} \Phi + r^2 \Phi v_r^2 - r \left(\Phi \lambda_r r + 3 r \Phi_r - 4 \Phi \right) v_r \right. \right. \right.$$

$$\left. \left. \left. + \left(-r^2 \Phi_r + 4 \Phi r \right) \lambda_r + 4 r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (48 \epsilon + 48) \pi r^2 \right) \right\}$$

>

▼ **d) An exact solution for the nonlinear system of differential equations conformed by the components of W_{α}^{β}**

Create an ODE system with the nonzero components of W_{μ}^{ν}

> $ode_{system} := \text{map}(u \rightarrow \text{rhs}(u) = 0, \text{rhs}((191)))$

$$ode_{system} := \left\{ \frac{1}{8 r^2} \left(\left(2 r^2 \Phi_{r,r} - 2 r^2 v_{r,r} \Phi + \left(r^2 v_r - r^2 \lambda_r - 4 r \right) \Phi_r \right. \right. \right. \quad (192)$$

$$\left. \left. \left. - \Phi \left(v_r^2 r^2 - v_r \lambda_r r^2 - 4 \right) \right) e^{-\lambda} - 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right) = 0, \right.$$

$$\begin{aligned}
& \frac{1}{8r^2} \left(\left(-6r^2 \Phi_{r,r} + 2r^2 v_{r,r} \Phi + r^2 \Phi v_r^2 - r (\Phi \lambda_r r - r \Phi_r + 4 \Phi) v_r \right. \right. \\
& \left. \left. + (3r^2 \Phi_r - 4 \Phi r) \lambda_r + 4r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (-16 \epsilon - 16) \pi r^2 \right) \\
& = 0, \frac{1}{8r^2} \left(\left(2r^2 \Phi_{r,r} + 2r^2 v_{r,r} \Phi + r^2 \Phi v_r^2 - r (\Phi \lambda_r r + 3r \Phi_r - 4 \Phi) v_r \right. \right. \\
& \left. \left. + (-r^2 \Phi_r + 4 \Phi r) \lambda_r + 4r \Phi_r - 4 \Phi \right) e^{-\lambda} + 4 \Phi + (48 \epsilon + 48) \pi r^2 \right) = 0 \}
\end{aligned}$$

Run a differential elimination process towards identifying singular cases, frequently simpler to solve: there are three cases

> *cases* := [PDEtools:-casesplit(*ode*_{system}, *caseplot*)] :

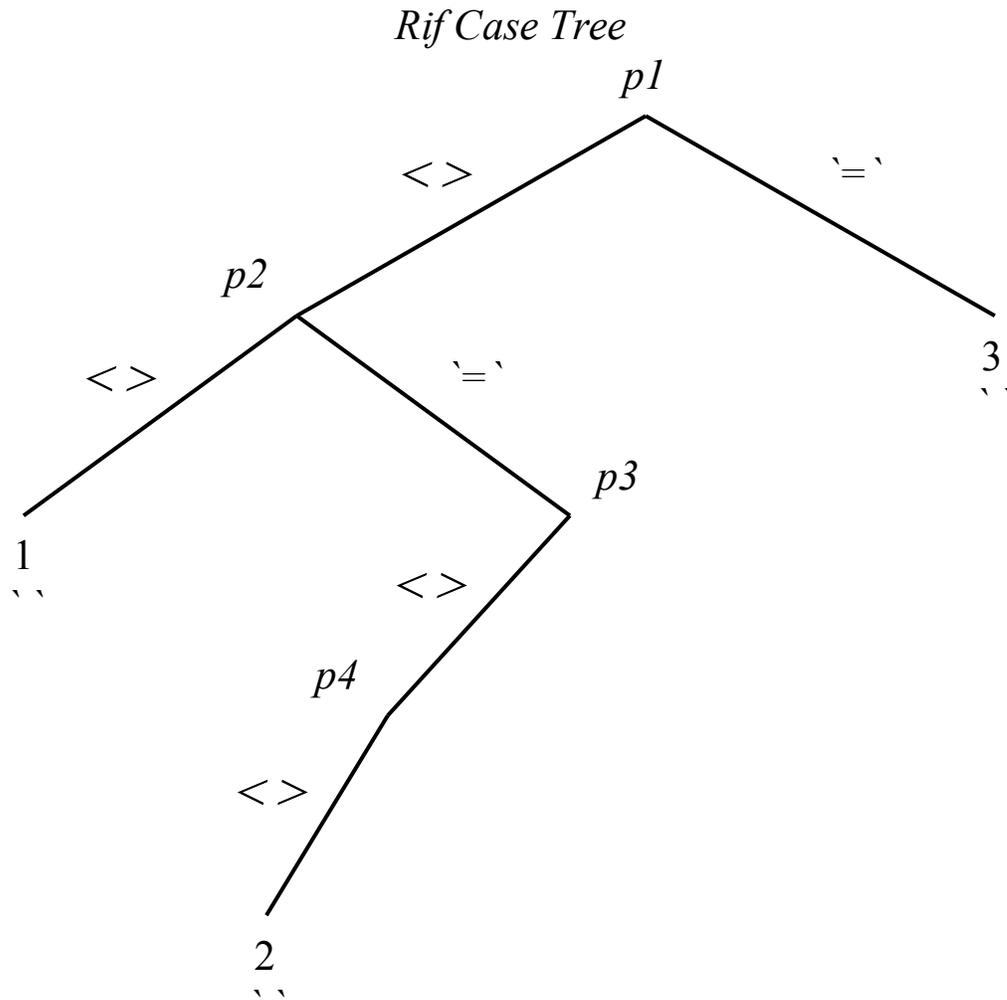
===== *Pivots Legend* =====

$$p1 = -r \Phi_r + 2 \Phi$$

$$\begin{aligned}
p2 = & r^2 \Phi (v_r r - 2) \Phi_{r,r} - (r^2 v_{r,r} \Phi + (-r^2 v_r + 2r) \Phi_r + \Phi (v_r^2 r^2 \\
& - 2)) (r \Phi_r - 2 \Phi)
\end{aligned}$$

$$p3 = \Phi + (12 \epsilon + 12) r^2 \pi$$

$$\begin{aligned}
p4 = & -12r \left(\frac{\Phi}{12} + \pi r^2 (\epsilon + 1) \right) \Phi_r + 2 \Phi^2 + 28 \pi r^2 (\epsilon + 1) \Phi + 32 \pi^2 r^4 (\epsilon \\
& + 1)^2
\end{aligned}$$



Check the length of each of these three cases:

> `map(length, cases)`

[5399, 1661, 405]

(193)

So the third one, a singular case, is of reasonably small size ...

> `sys[3] := op(1, cases[3])`

$$\text{sys}_3 := \left[e^{-\lambda} = -\frac{4 \pi r^2 (\epsilon + 1)}{\Phi}, \lambda_r = 0, v_{r,r} \right]$$

(194)

$$= \left[\frac{-r^4 \pi (\epsilon + 1) v_r^2 + 2 r^3 \pi (\epsilon + 1) v_r + \Phi + (4 \epsilon + 4) \pi r^2}{2 r^4 \pi (\epsilon + 1)}, \Phi_r = \frac{2 \Phi}{r} \right]$$

Compute an exact solution for it

> `constraint, subsystem := selectremove(has, sys[3], exp)`

$$\text{constraint, subsystem} := \left[e^{-\lambda} = -\frac{4 \pi r^2 (\epsilon + 1)}{\Phi} \right], \left[\lambda_r = 0, v_{r,r} \right]$$

(195)

$$= \frac{-r^4 \pi (\epsilon + 1) v_r^2 + 2r^3 \pi (\epsilon + 1) v_r + \Phi + (4\epsilon + 4) \pi r^2}{2r^4 \pi (\epsilon + 1)}, \Phi_r = \frac{2\Phi}{r}$$

> $sol_{subsystem} := dsolve(subsystem, explicit)$

$$sol_{subsystem} := \left\{ \Phi = _C1 r^2, v = \right. \quad (196)$$

$$\left. -\frac{1}{\sqrt{\pi (\epsilon + 1)}} \left(\ln \left(\frac{(32\epsilon + 32) \pi + 4_C1}{\left(\pi (\epsilon + 1) \left(r \frac{\sqrt{(8\epsilon + 8) \pi + _C1}}{\sqrt{\pi (\epsilon + 1)}} _C2 - _C3 \right) \right)^2} \right) \right. \right.$$

$$\left. \sqrt{\pi (\epsilon + 1)} + \ln(r) \left(\sqrt{(8\epsilon + 8) \pi + _C1} - 2\sqrt{\pi (\epsilon + 1)} \right) \right), \lambda = _C2 \left. \right\}$$

Specialize one of these constants using the constraint

> $eval(constraint, sol_{subsystem})$

$$\left[e^{-_C2} = -\frac{4\pi (\epsilon + 1)}{_C1} \right] \quad (197)$$

> $solve((197), _C1)$

$$\left\{ _C1 = -\frac{4\pi (\epsilon + 1)}{e^{-_C2}} \right\} \quad (198)$$

The exact solution then is

> $solution := subs((198), sol_{subsystem})$

$$solution := \left\{ \Phi = -\frac{4\pi (\epsilon + 1) r^2}{e^{-_C2}}, v = \right. \quad (199)$$

$$\left. -\frac{1}{\sqrt{\pi (\epsilon + 1)}} \left(\ln \left(\frac{32 e^{-_C2} - 16}{\left(e^{-_C2} \left(r \frac{\sqrt{4} \sqrt{\frac{\pi (\epsilon + 1) (2 e^{-_C2} - 1)}{e^{-_C2}}}}{\sqrt{\pi (\epsilon + 1)}} _C2 - _C3 \right) \right)^2} \right) \right. \right.$$

$$\left. \sqrt{\pi (\epsilon + 1)} + \ln(r) \left(\sqrt{4} \sqrt{\frac{\pi (\epsilon + 1) (2 e^{-_C2} - 1)}{e^{-_C2}}} - 2\sqrt{\pi (\epsilon + 1)} \right) \right),$$

$$\lambda = _C2$$

Verifying this result

> odetest(solution, ode_{system})

{0}

(200)

>

▼ The Equivalence problem between two metrics

From the "[What is new in Physics in Maple 2016](#)" page:

In the Maple PDEtools package, you have the mathematical tools - including a complete symmetry approach - to work with the underlying [Einstein's] partial differential equations. [By combining that functionality with the one in the Physics and Physics:-Tetrads package] you can also formulate and, depending on the metrics also resolve, the equivalence problem; that is: to answer whether or not, given two metrics, they can be obtained from each other by a transformation of coordinates, as well as compute the transformation.

▼ Example from: A. Karlhede, "A Review of the Geometrical Equivalence of Metrics in General Relativity", *General Relativity and Gravitation, Vol. 12, No. 9, 1980*

> restart,

with(Physics) : with(Tetrads) : Setup(auto = true, tetradmetric = null, signature = `+---`)

Setting lowercaselatin_ah letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a, \mu}, \eta_{a, b}, \gamma_{a, b, c}, \lambda_{a, b, c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism

(see ?Physics,tetrads), $l_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu}$

** Partial match of 'auto' against keyword 'automaticsimplication'*

[automaticsimplication = true, signature = + ---, tetradmetric = {(1, 2) = 1, (3, 4) = -1}] (201)

To formulate the problem, set first some symbols to represent the changed metric, changed mass and changed coordinates - no mathematics at this point

> mt, tt, rt, thetat, phit := m, t, r, \vartheta, \varphi

mt, tt, rt, thetat, phit := m, t, r, \vartheta, \varphi

(202)

Set now a new coordinates system, call it Y, involving the *new coordinates* (in the paper they are represented with a tilde on top of the letters)

> Coordinates ($Y = [tt, rt, thetat, phit]$)

Default differentiation variables for d_- , D_- and dA lembertian are: $\{Y = (t, r, \vartheta, \varphi)\}$

Systems of spacetime Coordinates are: $\{Y = (t, r, \vartheta, \varphi)\}$

$$\{Y\} \quad (203)$$

According to eq.(7.6) of the paper, the line element of Schwarzschild solution in isotropic spherical coordinates is given by

$$\begin{aligned} > ds^2 := & \left(\frac{\left(1 - \frac{mt}{2rt}\right)}{\left(1 + \frac{mt}{2rt}\right)} \right)^2 d_-(tt)^2 - \left(1 + \frac{mt}{2rt}\right)^4 \cdot (d_-(rt)^2 + rt^2 d_-(thetat)^2 \\ & + rt^2 \sin(thetat)^2 \cdot d_-(phit)^2) \\ ds^2 := & \frac{(-2r + m)^2 \partial(t)^2}{(2r + m)^2} \\ & - \frac{(2r + m)^4 (\partial(r)^2 + r^2 \partial(\vartheta)^2 + r^2 \sin(\vartheta)^2 \partial(\varphi)^2)}{16r^4} \end{aligned} \quad (204)$$

Set this to be the metric

> Setup(metric = ds^2) :

> g_[]

$$g_{\mu, \nu} = \begin{bmatrix} \frac{(-2r + m)^2}{(2r + m)^2} & 0 & 0 & 0 \\ 0 & -\frac{(2r + m)^4}{16r^4} & 0 & 0 \\ 0 & 0 & -\frac{(2r + m)^4}{16r^2} & 0 \\ 0 & 0 & 0 & -\frac{(2r + m)^4 \sin(\vartheta)^2}{16r^2} \end{bmatrix} \quad (205)$$

In connection with the transformation used further below, compute now the Petrov type and the Weyl scalars for this metric, just to have an idea of what is behind this metric.

> PetrovType()

$$\text{"D"} \quad (206)$$

> Weyl[scalars]

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = -\frac{64r^3 m}{(2r + m)^6}, \Psi_3 = 0, \Psi_4 = 0 \quad (207)$$

We see that the Weyl scalars are already in canonical form, only $\Psi_2 \neq 0$ and the important thing: it depends on only one coordinate, r .

We want to see if this metric (205) is equivalent to Schwarzschild metric in standard spherical coordinates

> `g_[sc]`

Systems of spacetime Coordinates are: $\{X = (t, r, \theta, \phi), Y = (t, r, \vartheta, \varphi)\}$

Warning, changing the differentiation variables used to compute the Christoffel

symbols from $[t, r, \vartheta, \varphi]$ to $[t, r, \theta, \phi]$ while the spacetime metric depends on $[r, \vartheta, m]$

Default differentiation variables for d_-, D_- and $dAlembertian$ are: $\{X = (t, r, \theta, \phi)\}$

The Schwarzschild metric in coordinates $[t, r, \theta, \phi]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r - 2m}{r} & 0 & 0 & 0 \\ 0 & \frac{r}{-r + 2m} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin(\theta)^2 \end{bmatrix} \quad (208)$$

The equivalence we want to resolve is regarding an arbitrary relationship $m(m)$ between the masses used in (205) and (208) and a generic change of variables from X to Y

> `TR := {phi = Phi(Y), r = R(Y), t = T(Y), theta = Theta(Y)}`

$$TR := \{\phi = \Phi(Y), r = R(Y), t = T(Y), \theta = \Theta(Y)\} \quad (209)$$

> `CompactDisplay(TR)`

$\Phi(t, r, \vartheta, \varphi)$ will now be displayed as Φ

$R(t, r, \vartheta, \varphi)$ will now be displayed as R

$T(t, r, \vartheta, \varphi)$ will now be displayed as T

$\Theta(t, r, \vartheta, \varphi)$ will now be displayed as Θ (210)

Using a differential equation mindset, **the formulation** of the equivalence between (208) and (205) under the transformation (209) is actually simple: change variables in (208), using (209) and the [Physics:-TransformCoordinates](#) command (this is the command that changes variables in tensorial expressions), then equate the result to (205), then try to solve the problem for the unknowns $m(m)$, $\Phi(Y)$, $R(Y)$, $\Theta(Y)$ and $T(Y)$.

We note at this point, however, that the Weyl scalars for Schwarzschild metric in this standard form (208) are also in canonical form of Petrov type D and also depend on only one variable, r

> `PetrovType()`

"D" (211)

> `Weyl[scalars]`

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = -\frac{m}{r^3}, \Psi_3 = 0, \Psi_4 = 0 \quad (212)$$

The fact that the Weyl scalars in both cases ((207) and (212)) are in canonical form (only $\Psi_2 \neq 0$) and in both cases this scalar depends on only one coordinate is already an indicator

that the transformation involved changes only one variable in terms of the other one. So one could just search for a transformation of the form $r = R(r)$ and resolve the problem instantly.

> *TransformCoordinates*($r = R(r)$, $g_{[\mu, \nu]}$)

$$\left[\begin{array}{cccc} \frac{R(r) - 2m}{R(r)} & 0 & 0 & 0 \\ 0 & \frac{R_r^2 R(r)}{-R(r) + 2m} & 0 & 0 \\ 0 & 0 & -R(r)^2 & 0 \\ 0 & 0 & 0 & -R(r)^2 \sin(\vartheta)^2 \end{array} \right] \quad (213)$$

> *convert*(*rhs*((205)) = (213), *setofequations*)

$$\left\{ 0 = 0, \frac{(-2r + m)^2}{(2r + m)^2} = \frac{R(r) - 2m}{R(r)}, -\frac{(2r + m)^4}{16r^4} = \frac{R_r^2 R(r)}{-R(r) + 2m}, \right. \quad (214)$$

$$\left. \left\{ \frac{(2r + m)^4}{16r^2} = -R(r)^2, -\frac{(2r + m)^4 \sin(\vartheta)^2}{16r^2} = -R(r)^2 \sin(\vartheta)^2 \right. \right\},$$

> *pdsolve*((214), [R, mt])

$$\left\{ m = m, R(r) = \frac{(2r + m)^2}{4r} \right\}, \left\{ m = -m, R(r) = -\frac{(m - 2r)^2}{4r} \right\} \quad (215)$$

To make the problem slightly more general, consider instead a generic transformation for r in terms of all of $Y = (t, r, \vartheta, \varphi)$, and also allow the time to change, so we search for two transformation functions resolving the equivalence

> $tr := \text{select}(\text{has}, TR, [r, t])$

$$tr := \{r = R, t = T\} \quad (216)$$

> *CompactDisplay*((216))

$R(t, r, \vartheta, \varphi)$ will now be displayed as R

$T(t, r, \vartheta, \varphi)$ will now be displayed as T (217)

Transform the coordinates in the metric

> *TransformCoordinates*($tr, g_{[\mu, \nu]}$)

$$\left[\left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_t^2 + R_t^2 R^2}{R(-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_r T_t + R_r R_t R^2}{R(-R + 2m)}, \right. \right. \quad (218)$$

$$\left. \left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\vartheta T_t + R_\vartheta R_t R^2}{R(-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi T_t + R_\varphi R_t R^2}{R(-R + 2m)}, \right] \right]$$

$$\left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_r T_t + R_r R_t R^2}{R(-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_r^2 + R_r^2 R^2}{R(-R + 2m)}, \right]$$

$$\begin{aligned}
& \left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\vartheta} T_r + R_{\vartheta} R_r R^2}{R (-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\varphi} T_r + R_{\varphi} R_r R^2}{R (-R + 2m)}, \right. \\
& \left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\vartheta} T_t + R_{\vartheta} R_t R^2}{R (-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\vartheta} T_r + R_{\vartheta} R_r R^2}{R (-R + 2m)}, \right. \\
& \left. \frac{T_{\vartheta}^2 (R - 2m)}{R} - \frac{R_{\vartheta}^2 R}{R - 2m} - R^2, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\varphi} T_{\vartheta} + R_{\varphi} R_{\vartheta} R^2}{R (-R + 2m)} \right], \\
& \left[\frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\varphi} T_t + R_{\varphi} R_t R^2}{R (-R + 2m)}, \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\varphi} T_r + R_{\varphi} R_r R^2}{R (-R + 2m)}, \right. \\
& \left. \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_{\varphi} T_{\vartheta} + R_{\varphi} R_{\vartheta} R^2}{R (-R + 2m)}, \frac{1}{R (-R + 2m)} \left(-4 \left(-\frac{R}{2} \right. \right. \right. \\
& \left. \left. \left. + m \right)^2 T_{\varphi}^2 + 2 \left(\frac{R_{\varphi}^2}{2} + (\cos(\vartheta) + 1) R \left(-\frac{R}{2} + m \right) (\cos(\vartheta), -1) \right) R^2 \right) \right]
\end{aligned}$$

Change also the relationship between the masses so that $m(m) \neq m$, for instance:

$$> \text{subs} \left(mt = \frac{1}{mt^2}, \text{(205)} \right)$$

$$g_{\mu, \nu} = \left[\left[\frac{\left(-2r + \frac{1}{m^2} \right)^2}{\left(2r + \frac{1}{m^2} \right)^2}, 0, 0, 0 \right], \right.$$

(219)

$$\left[0, -\frac{\left(2r + \frac{1}{m^2} \right)^4}{16r^4}, 0, 0 \right],$$

$$\left[0, 0, -\frac{\left(2r + \frac{1}{m^2} \right)^4}{16r^2}, 0 \right],$$

$$\left[0, 0, 0, -\frac{\left(2r + \frac{1}{m^2} \right)^4 \sin(\vartheta)^2}{16r^2} \right]$$

> $\text{convert}(\text{rhs}(\text{(219)}) = \text{(218)}, \text{setofequations})$

$$\left\{ \begin{aligned}
0 &= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_r T_t + R_r R_t R^2}{R (-R + 2m)}, 0 = \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi T_r + R_\varphi R_r R^2}{R (-R + 2m)}, 0 \quad (220) \\
&= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi T_t + R_\varphi R_t R^2}{R (-R + 2m)}, 0 \\
&= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi T_\vartheta + R_\varphi R_\vartheta R^2}{R (-R + 2m)}, 0 \\
&= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\vartheta T_r + R_\vartheta R_r R^2}{R (-R + 2m)}, 0 = \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_\vartheta T_t + R_\vartheta R_t R^2}{R (-R + 2m)}, \\
\frac{\left(-2r + \frac{1}{m^2} \right)^2}{\left(2r + \frac{1}{m^2} \right)^2} &= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_t^2 + R_t^2 R^2}{R (-R + 2m)}, -\frac{\left(2r + \frac{1}{m^2} \right)^4}{16r^4} \\
&= \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_r^2 + R_r^2 R^2}{R (-R + 2m)}, -\frac{\left(2r + \frac{1}{m^2} \right)^4}{16r^2} = \frac{T_\vartheta^2 (R - 2m)}{R} \\
&- \frac{R_\vartheta^2 R}{R - 2m} - R^2, -\frac{\left(2r + \frac{1}{m^2} \right)^4 \sin^2(\vartheta)}{16r^2} = \frac{1}{R (-R + 2m)} \left(-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi^2 + 2 \left(\frac{R_\varphi^2}{2} + (\cos(\vartheta) + 1) R \left(-\frac{R}{2} + m \right) (\cos(\vartheta) - 1) \right) R^2 \right)
\end{aligned} \right.$$

This problem, shown in Karlhede's paper as the example of the approach he summarized, is solvable using only differential elimination, in no time, obtaining the same solution shown in the paper with equation number (7.10)

> *PDEtools:-casesplit*((220), [R, T, mt])

$$\left[R = -\frac{(m - 2r)^2}{4r}, T_t = -1, T_r = 0, T_\vartheta = 0, T_\varphi = 0, m^2 = -\frac{1}{m} \right] \&\text{where } [m \neq 0], \quad (221)$$

$$\left[R = -\frac{(m-2r)^2}{4r}, T_t = 1, T_r = 0, T_\vartheta = 0, T_\varphi = 0, m^2 = -\frac{1}{m} \right] \&\text{where } [m \neq 0],$$

$$\left[R = \frac{(2r+m)^2}{4r}, T_t = -1, T_r = 0, T_\vartheta = 0, T_\varphi = 0, m^2 = \frac{1}{m} \right] \&\text{where } [m \neq 0],$$

$$\left[R = \frac{(2r+m)^2}{4r}, T_t = 1, T_r = 0, T_\vartheta = 0, T_\varphi = 0, m^2 = \frac{1}{m} \right] \&\text{where } [m \neq 0]$$

> `pdsolve((221)[1], [R, T, mt])`

$$\left\{ m = -\frac{1}{\sqrt{-m}}, R = -\frac{(m-2r)^2}{4r}, T = -t + _C2 \right\}, \left\{ m = \frac{1}{\sqrt{-m}}, R = -\frac{(m-2r)^2}{4r}, T = -t + _C1 \right\} \quad (222)$$

The fact that the time t appears defined in terms of the transformed time $T(Y) = -t + _CI$ involving an arbitrary constant is expected: the time does not enter the metric, it only enters through derivatives of $T(Y)$ entering the Jacobian of the transformation used to change variables in tensorial expressions (the metric) in (218).

Summary: the approach shown above, based on formulating the problem for the transformation functions of the equivalence and solving for them the differential equations using the commands in PDEtools, after restricting the generality of the transformation functions by looking at the form of the Weyl scalars, works well for other cases too, specially now that, in Maple 2016, the Weyl scalars can be expressed also in canonical form in one go (see previous [Mapleprimes post on "Tetrads and Weyl scalars in canonical form"](#)). Also important: in Maple 2016 it is present the functionality necessary to implement the approach of section 9.2 of the Exact solutions book as well.

>

▼ **Equivalence for Schwarzschild metric (spherical and Krustal coordinates)*

This problem is interesting because:

- It is well known in the literature
- It involves departing from a metric expressed in "mixed coordinates"
- When writing the metric entirely in Krustal coordinates, the dependence involves special functions (LambertW)

▼ *Formulation of the problem (remove mixed coordinates)*

> `restart;`

> `with(Physics) : with(Tetrads) : Setup(auto = true);`

Setting lowercaselatin_ah letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism

(see ?Physics,tetrads), $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$

* Partial match of 'auto' against keyword 'automaticsimplication'

[automaticsimplication = true]

(223)

The departure point, Schwarzschild metric in spherical coordinates

> g_[sc]

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: [m]

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (224)$$

Introduce now Krustal coordinates following the literature (see wikipedia) and the corresponding line element involving "mixed" coordinates

> Coordinates ($K = [u, \vartheta, \phi, v]$)

Systems of spacetime Coordinates are: $\{K = (u, \vartheta, \phi, v), X = (r, \theta, \phi, t)\}$

$\{K, X\}$

(225)

$$> ds^2 := \frac{16 (\partial(v)) m^2 e^{-\frac{r}{2m}} (\partial(u)) - ((1 - \cos(\vartheta))^2 (\partial(\phi))^2 + (\partial(\vartheta))^2) r^3}{r}$$

$$ds^2 := \frac{16 \partial(v) m^2 e^{-\frac{r}{2m}} \partial(u) - ((1 - \cos(\vartheta))^2 \partial(\phi)^2 + \partial(\vartheta)^2) r^3}{r} \quad (226)$$

The mixing of variables is visible: in the line element above is in Krustal coordinates but you also see r , which belongs to the X (not K) coordinates.

For the purpose of formulating problem free of this mixing of coordinates, set the metric now to be (226)

> Setup(diff = [K], metric = (226), quiet) :

> g_[]

(227)

$$g_{\mu, \nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{8 m^2 e^{-\frac{r}{2m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\vartheta)^2 & 0 \\ \frac{8 m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & 0 \end{bmatrix} \quad (227)$$

To remove the mix of coordinates, introduce a transformation with unknown transformation functions $\{f, h\}$, change variables, and resolve for the transformation functions $\{f, g\}$ (this in itself is resolving a form of *equivalence* problem).

$$> tr_0 := \{u = f(r, t), v = h(r, t)\}$$

$$tr_0 := \{u = f(r, t), v = h(r, t)\} \quad (228)$$

$$> CompactDisplay(tr_0)$$

f(r, t) will now be displayed as f

h(r, t) will now be displayed as h

$$(229)$$

$$> TransformCoordinates(tr_0, g_{[\mu, \nu]})$$

$$\begin{bmatrix} \frac{16 f_r h_r m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & \frac{8 m^2 e^{-\frac{r}{2m}} (h_r f_t + f_r h_t)}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ \frac{8 m^2 e^{-\frac{r}{2m}} (h_r f_t + f_r h_t)}{r} & 0 & 0 & \frac{16 f_t h_t m^2 e^{-\frac{r}{2m}}}{r} \end{bmatrix} \quad (230)$$

Equate to (224) and solve

$$> convert((230) = rhs((224)), setofequations)$$

$$\left\{ 0 = 0, -r^2 = -r^2, -r^2 \sin(\theta)^2 = -r^2 \sin(\theta)^2, \frac{8 m^2 e^{-\frac{r}{2m}} (h_r f_t + f_r h_t)}{r} = 0, \right. \quad (231)$$

$$\left. \left\{ \frac{16 f_r h_r m^2 e^{-\frac{r}{2m}}}{r} = \frac{r}{-r + 2m}, \frac{16 f_t h_t m^2 e^{-\frac{r}{2m}}}{r} = \frac{r - 2m}{r} \right\} \right.$$

$$> pdsolve((231))$$

$$\left\{ f = _C1 + _C2 \sqrt{r - 2m} e^{\frac{r}{4m}} e^{-\frac{t}{4m}}, h = -\frac{e^{\frac{r+t}{4m}} \sqrt{r - 2m}}{_C2} + _C3 \right\}, \left\{ f \right. \quad (232)$$

$$= \left. \begin{aligned} & _C1 + _C2 \sqrt{r-2m} e^{\frac{r}{4m}} e^{\frac{t}{4m}}, h = -\frac{\sqrt{r-2m} e^{\frac{r-t}{4m}}}{_C2} + _C3 \end{aligned} \right\}$$

Without loss of generality, set $[_C1 = 0, _C2 = 1, _C3 = 0]$

> $tr := combine(subs([_C1 = 0, _C2 = 1, _C3 = 0], eval((228), (232)[1])))$

$$tr := \left\{ u = \sqrt{r-2m} e^{\frac{r-t}{4m}}, v = -e^{\frac{r+t}{4m}} \sqrt{r-2m} \right\} \quad (233)$$

Check it out:

> $g_[]$

$$g_{\mu, \nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{8m^2 e^{-\frac{r}{2m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\vartheta)^2 & 0 \\ \frac{8m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & 0 \end{bmatrix} \quad (234)$$

> $TransformCoordinates(tr, g_[\mu, \nu], [X], [K])$

$$\begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad (235)$$

Here is where things become computationally challenging: compute the inverse of the transformation (233)

> $itr := simplify(normal(solve((233), \{r, t\}), expanded))$

Warning, solve may be ignoring assumptions on the input variables.

$$itr := \left\{ r = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, t = 2 \ln \left(-\frac{v}{u} \right) m \right\} \quad (236)$$

This itr involves the [LambertW function](#). Set now the metric to be the standard Schwarzschild's metric in spherical coordinates (224) and compute use itr to get the form of the metric entirely in Krustal coordinates - no more mixings

> $g_ [sc]$

Warning, changing the differentiation variables used to compute the Christoffel

symbols from $[u, \vartheta, \phi, v]$ to $[r, \theta, \phi,$

$t]$ while the spacetime metric depends on $[\vartheta, m, r]$

Default differentiation variables for d_- , D_- and $dAlembertian$ are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (237)$$

So this is Schwarzschild's solution all in Krustal coordinates

> *TransformCoordinates* (*itr*, $g_{[\mu, \nu]}$, $[K]$, $[X]$)

$$\left[\begin{array}{l} \left[0, 0, 0, -\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v} \right] \\ \left[0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2, 0, 0 \right] \\ \left[0, 0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2 \sin^2(\vartheta), 0 \right] \\ \left[-\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v}, 0, 0, 0 \right] \end{array} \right], \quad (238)$$

This metric involves the LambertW function in a non-simplifiable form (to avoid that is the reason for people to use the mixed coordinates version (227)).

>

▼ Solving the Equivalence

We now have the two forms: (235) in spherical and (238) in Krustal coordinates, so we can formulate the equivalence problem from one coordinate system to the other one.

The transformation to be resolved does not need to involve ϕ because neither ϕ nor φ enter either of the two metrics.

The transformation does not need to involve θ or ϑ because they enter the metrics in exactly the same position and with the same dependence.

In addition the Weyl scalars of both metrics are in canonical form and the only scalar

different from zero, that is Ψ_2 does not depend on any of $\{\phi, \theta, \varphi, \vartheta\}$

So we look for a generic transformation from spherical to Krustal of the form

$$\begin{aligned} > \{r = R(K), t = T(K)\} \\ & \qquad \qquad \qquad \{r = R(K), t = T(K)\} \end{aligned} \quad (239)$$

> *CompactDisplay*((239))

$$\begin{aligned} & R(u, \vartheta, \varphi, v) \text{ will now be displayed as } R \\ & T(u, \vartheta, \varphi, v) \text{ will now be displayed as } T \end{aligned} \quad (240)$$

The metric set in this moment is in spherical coordinates, (237), so change using (239) and equate to (238) in Krustal coordinates

> *convert*(*TransformCoordinates*((239), $g_{[\mu, \nu]}$, $[K]$, $[X]$) = (238),
setofequations)

$$\left\{ \begin{aligned} & \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_u^2 + R_u^2 R^2}{(-R + 2m) R} = 0, \quad \frac{-4 \left(-\frac{R}{2} + m \right)^2 T_v^2 + R_v^2 R^2}{(-R + 2m) R} = 0, \end{aligned} \right. \quad (241)$$

$$\begin{aligned} & \frac{1}{(-R + 2m) R} \left(-4 \left(-\frac{R}{2} + m \right)^2 T_\varphi^2 + 2 R^2 \left(\frac{R_\varphi^2}{2} + \left(-\frac{R}{2} \right. \right. \right. \\ & \left. \left. \left. + m \right) (\cos(\vartheta) + 1) R (\cos(\vartheta) - 1) \right) \right) = -4 \left(W \left(-\frac{u v e^{-1}}{2m} \right) \right. \\ & \left. + 1 \right)^2 m^2 \sin(\vartheta)^2, \quad \frac{-4 T_v \left(-\frac{R}{2} + m \right)^2 T_u + R_v R_u R^2}{(-R + 2m) R} = \\ & -\frac{8 W \left(-\frac{u v e^{-1}}{2m} \right) m^2}{\left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) u v}, \quad \frac{-4 T_v \left(-\frac{R}{2} + m \right)^2 T_\varphi + R_v R_\varphi R^2}{(-R + 2m) R} = 0, \\ & \frac{-4 T_v \left(-\frac{R}{2} + m \right)^2 T_\vartheta + R_v R_\vartheta R^2}{(-R + 2m) R} = 0, \end{aligned}$$

$$\frac{-4 T_{\phi} \left(-\frac{R}{2} + m \right)^2 T_u + R_{\phi} R_u R^2}{(-R + 2m) R} = 0,$$

$$\frac{-4 T_{\phi} \left(-\frac{R}{2} + m \right)^2 T_{\vartheta} + R_{\phi} R_{\vartheta} R^2}{(-R + 2m) R} = 0,$$

$$\frac{-4 T_{\vartheta} \left(-\frac{R}{2} + m \right)^2 T_u + R_{\vartheta} R_u R^2}{(-R + 2m) R} = 0, -\frac{R_{\vartheta}^2 R}{R - 2m} - R^2$$

$$+ \frac{T_{\vartheta}^2 (R - 2m)}{R} = -4 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right)^2 m^2 ,$$

Again, this is a nonlinear, non-rational PDE system in two unknowns depending on two independent variables (see (239)). You can now either call pdsolve on (241), solving the problem in one step, or first split into cases without solving any differential equation, just doing differential elimination, to see the cases

> PDEtools:-casesplit((241))

$$\left[T_u = \frac{2m}{u}, T_{\vartheta} = 0, T_{\phi} = 0, T_v = -\frac{2m}{v}, R = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m \right] \&\text{where (242)}$$

$$[], \left[T_u = -\frac{2m}{u}, T_{\vartheta} = 0, T_{\phi} = 0, T_v = \frac{2m}{v}, R = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m \right]$$

&where []

So by only using differential elimination we removed all nonlinearities. This problem is actually easy for the differential equation routines

> pdsolve((241))

$$\left\{ R = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, T = -2m \ln(u) + 2m \ln(v) + _CI \right\}, \left\{ R \right. \quad (243)$$

$$\left. = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, T = 2m \ln(u) - 2m \ln(v) + _CI \right\}$$

So the transformation of coordinates resolving the equivalence between (235) and (238) is

> eval((239), (243)[1])

$$\left\{ r = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, t = -2m \ln(u) + 2m \ln(v) + _CI \right\} \quad (244)$$

Check it transforming (235) fully written in spherical coordinates into (238) fully written in Krustal coordinates

> g_[]

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (245)$$

> TransformCoordinates ((244), g_[mu, nu], [K], [X])

$$\begin{bmatrix} \left[0, 0, 0, -\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v} \right], \\ \left[0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2, 0, 0 \right], \\ \left[0, 0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2 \sin(\theta)^2, 0 \right], \\ \left[-\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v}, 0, 0, 0 \right] \end{bmatrix}, \quad (246)$$

>

▼ *On the 3+1 split of the 4D Einstein equations

Consider the Lemaitre-Tolman-Bondi metric.

$$g_{\mu, \nu} = \begin{bmatrix} \frac{\left(\frac{\partial}{\partial r} R(t, r)\right)^2}{1 + 2 E(r)} & 0 & 0 & 0 \\ 0 & R(t, r)^2 & 0 & 0 \\ 0 & 0 & R(t, r)^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and the EnergyMomentum tensor

$$T^{\mu}_{\nu} = -\rho_M(t, r) \delta_4^{\mu} \delta_{\nu}^4 - \rho_{\Lambda} \delta_{\nu}^{\mu}$$

a) Show that the relation between the matter density $\rho_M(t, r)$, the vacuum energy $\rho_\Lambda = \text{constant}$ and

the the gravitational mass $M(r) = \frac{R(t, r) \left(\left(\frac{\partial}{\partial t} R(t, r) \right)^2 - 2 E(r) \right)}{2}$ of a comoving sphere of radius r is given by (see wikipedia)

$$\rho_M + \rho_\Lambda = \frac{M_r}{4 R^2 R_r \pi}$$

b) Show that the 4D Einstein equations and their 3 + 1 split in terms of the extrinsic curvature are the same equations

▼ Solution

Formulation of the problem

Load the ThreePlusOne Physics package and the Lemaitre-Tolman-Bondi metric, that in the Maple database of solutions to Einstein's equations can be retrieved directly using a portion of the word Tolman as an index to the metric `g`

> restart, with(Physics) : with(ThreePlusOne)
Setting lowercaselatin_is letters to represent space indices

Defined as 4D, spacetime tensors that are purely spatial (see ?Physics, ThreePlusOne),

$$\gamma_{\mu, \nu}, \mathcal{D}_\mu, \Gamma_{\mu, \nu, \alpha}, R_{\mu, \nu}, R_{\mu, \nu, \alpha, \beta}, \beta_{\mu'} n_{\mu'} t_{\mu'}, \mathbf{K}_{\mu, \nu}$$

Changing the signature of spacetime from (- - - +) to (+ + + -) in order to match the signature customarily used in the ADM formalism

[ADMEquations, Christoffel3, D3_, ExtrinsicCurvature, Lapse, Ricci3, Riemann3, Shift, TimeVector, UnitNormalVector, gamma3_] (247)

> g_tol

Systems of spacetime Coordinates are: {X = (r, \theta, \phi, t)}

Default differentiation variables for d_, D_ and dAlembertian are: {X = (r, \theta, \phi, t)}

The Tolman metric in coordinates [r, \theta, \phi, t]

Parameters: [R(t, r), E(r)]

$$g_{\mu, \nu} = \begin{bmatrix} \frac{\left(\frac{\partial}{\partial r} R(t, r) \right)^2}{1 + 2 E(r)} & 0 & 0 & 0 \\ 0 & R(t, r)^2 & 0 & 0 \\ 0 & 0 & R(t, r)^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (248)$$

> CompactDisplay((248))

E(r) will now be displayed as E

(248)

$$R(t, r) \text{ will now be displayed as } R \quad (249)$$

The [EnergyMomentum](#) tensor is related to Einstein's tensor by

> [EnergyMomentum](#) [definition]

$$T_{\mu, \nu} = \frac{G_{\mu, \nu}}{8 \pi} \quad (250)$$

and for this metric it is defined as

$$\begin{aligned} > \text{EnergyMomentum}_{\sim mu, \nu} = -\rho_M(t, r) \text{KroneckerDelta}_{\sim mu, 0} \text{KroneckerDelta}_{\nu, \sim 0} \\ & - \rho_{\Lambda} \text{KroneckerDelta}_{\sim mu, \nu} \\ T_{\nu}^{\mu} &= -\rho_M(t, r) \delta_4^{\mu} \delta_{\nu}^4 - \rho_{\Lambda} \delta_{\nu}^{\mu} \end{aligned} \quad (251)$$

where $\rho_M(t, r)$ is the matter density, $u^{\mu} = \delta_0^{\mu}$ is the 4-velocity of the matter that is comoving and we keep the vacuum energy $\rho_{\Lambda} = \text{constant}$ for illustration purposes only, and

> [CompactDisplay](#)((251))

$$\rho_M(t, r) \text{ will now be displayed as } \rho_M \quad (252)$$

> [Define](#)((251))

Defined objects with tensor properties

$$\left\{ \mathcal{D}_{\mu}, \mathcal{D}_{\mu}, \gamma_{\mu}, \sigma_{\mu}, R_{\mu, \nu}, R_{\mu, \nu}, R_{\mu, \nu, \alpha, \beta}, \mathbf{B}_{\mu}, C_{\mu, \nu, \alpha, \beta}, X_{\mu}, \partial_{\mu}, g_{\mu, \nu}, \mathbf{Y}_{\mu, \nu}, \gamma_{i, j}, \Gamma_{\mu, \nu, \alpha}, \Gamma_{\mu, \nu, \alpha}, G_{\mu, \nu}, T_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}, R_{\mu, \nu, \alpha, \beta}, t_{\mu}, \mathbf{K}_{\mu, \nu}, n_{\mu} \right\} \quad (253)$$

a) The relationship between the matter density $\rho_M(t, r)$, the vacuum energy $\rho_{\Lambda} = \text{constant}$ and the the gravitational mass $M(r)$

Take now the components of the 4D form of Einstein's equations (250) and derive an expression for $\rho_M(t, r)$ as a function of $R(t, r)$, $E(r)$

> [EQ4](#) := [TensorArray](#)((250))

$$\begin{aligned} EQ4 := & \left[\left[-\frac{\rho_{\Lambda} R_r^2}{1 + 2 E} = -\frac{R_r^2 (R_t^2 + 2 R R_{t, t} - 2 E)}{8 R^2 (1 + 2 E) \pi}, 0 = 0, 0 = 0, 0 = 0 \right], \right. \\ & \left[0 = 0, -\rho_{\Lambda} R^2 = -\frac{R (R_t R_{r, t} + R_{r, t, t} R + R_{t, t} R_r - E_r)}{8 R_r \pi}, 0 = 0, 0 = 0 \right], \\ & \left[0 = 0, 0 = 0, -\rho_{\Lambda} R^2 \sin(\theta)^2 = \right. \\ & \left. -\frac{R \sin(\theta)^2 (R_t R_{r, t} + R_{r, t, t} R + R_{t, t} R_r - E_r)}{8 R_r \pi}, 0 = 0 \right], \end{aligned} \quad (254)$$

$$\left[0 = 0, 0 = 0, 0 = 0, \rho_M + \rho_\Lambda = \frac{R_t^2 R_r + 2 R_t R_{r,t} R - 2 E_r R - 2 R_r E}{8 R_r R^2 \pi} \right]$$

Introduce $M(r)$, the gravitational mass of a sphere at radius r (see wikipedia for definitions)

$$\begin{aligned} > M(r) = -\frac{1}{2} \left(-\left(\frac{\partial}{\partial t} R(t, r) \right)^2 + 2 E(r) \right) R(t, r) \\ M(r) = -\left(-\frac{R_t^2}{2} + E \right) R \end{aligned} \quad (255)$$

The relationship we are looking for is in $EQ4_{4,4}$, so simplify the expression obtained for $\rho_M + \rho_\Lambda$ introducing $M(r)$ and eliminating $E(r)$ (see [simplify, siderelations](#))

> *simplify*($EQ4_{4,4}$, {(255)}, { $E(r)$ })

$$\rho_M + \rho_\Lambda = \frac{M_r}{4 R^2 R_r \pi} \quad (256)$$

b) Show that the 4D Einstein equations and their 3 + 1 split in terms of the extrinsic curvature are one and same system

Start from the active form of the ADM equations

> *eq* := *ADMEquations* (*inert* = *false*)

$$eq := \left[\left[\frac{(2 R_r R_t + R R_{r,t})^2}{R_r^2 R^2} - \mathbf{K}_{\alpha, \beta} \mathbf{K}^{\alpha, \beta} - \frac{4 (R_r E + E_r R)}{R_r R^2} = 16 \pi n_\alpha n_\beta T^{\alpha, \beta} \right] \right] \quad (257)$$

]

$$\left[\mathcal{D}_\beta (\mathbf{K}_\mu^\beta) - \mathbf{y}_\mu^\tau \left(-\frac{1}{R_r R} (2 (R_{r,r} \partial_\tau(r) + R_{r,t} \partial_\tau(t)) R_t \right. \right.$$

$$\left. \left. + 2 R_r (R_{r,t} \partial_\tau(r) + R_{t,t} \partial_\tau(t)) + \partial_\tau(R) R_{r,t} + R (R_{r,r,t} \partial_\tau(r) \right. \right.$$

$$\left. \left. + R_{r,t,t} \partial_\tau(t) \right) \right] + \frac{(2 R_r R_t + R R_{r,t}) (R_{r,r} \partial_\tau(r) + R_{r,t} \partial_\tau(t))}{R_r^2 R}$$

$$\left. \left. + \frac{(2 R_r R_t + R R_{r,t}) \partial_\tau(R)}{R_r R^2} \right) = -8 \pi \mathbf{y}_\mu^\beta n^\tau T_{\beta, \tau} \right]$$

$$\left[\mathbf{t}^\tau \mathcal{D}_\tau (\mathbf{K}_{\mu, \nu}) + \mathbf{K}_{\tau, \nu} \mathcal{D}_\mu (\mathbf{t}^\tau) + \mathbf{K}_{\mu, \tau} \mathcal{D}_\nu (\mathbf{t}^\tau) = -\frac{(2 R_r R_t + R R_{r,t}) \mathbf{K}_{\mu, \nu}}{R_r R} \right]$$

$$\begin{aligned}
& -2 \mathbf{K}_{\mu, \tau} \mathbf{K}_v^\tau + R_{\mu, v} - 8 \pi \left(\boldsymbol{\gamma}_\mu^\kappa \boldsymbol{\gamma}_v^\lambda T_{\kappa, \lambda} \right. \\
& \left. - \frac{\boldsymbol{\gamma}_{\mu, v} \left(\boldsymbol{\gamma}_\kappa^\lambda \boldsymbol{\gamma}^{\kappa, \sigma} T_{\lambda, \sigma} - T^{\alpha, \beta} n_\alpha n_\beta \right)}{2} \right) + \boldsymbol{\beta}^\tau \mathcal{D}_\tau(\mathbf{K}_{\mu, v}) + \mathbf{K}_{\tau, v} \mathcal{D}_\mu(\boldsymbol{\beta}^\tau) \\
& \left. + \mathbf{K}_{\mu, \tau} \mathcal{D}_v(\boldsymbol{\beta}^\tau) \right], \\
& \left[\boldsymbol{t}^\tau \mathcal{D}_\tau(\boldsymbol{\gamma}_{\mu, v}) + \boldsymbol{\gamma}_{\tau, v} \mathcal{D}_\mu(\boldsymbol{t}^\tau) + \boldsymbol{\gamma}_{\mu, \tau} \mathcal{D}_v(\boldsymbol{t}^\tau) = -2 \mathbf{K}_{\mu, v} + \boldsymbol{\beta}^\tau \mathcal{D}_\tau(\boldsymbol{\gamma}_{\mu, v}) \right. \\
& \left. + \boldsymbol{\gamma}_{\tau, v} \mathcal{D}_\mu(\boldsymbol{\beta}^\tau) + \boldsymbol{\gamma}_{\mu, \tau} \mathcal{D}_v(\boldsymbol{\beta}^\tau) \right]
\end{aligned}$$

The expression for $\rho_M + \rho_\Lambda$ in terms of $M(r)$ is obtained now from eq_1

> eq_1

$$\frac{(2 R_r R_t + R R_{r, t})^2}{R_r^2 R^2} - \mathbf{K}_{\alpha, \beta} \mathbf{K}^{\alpha, \beta} - \frac{4 (R_r E + E_r R)}{R_r R^2} = 16 \pi n_\alpha n_\beta T^{\alpha, \beta} \quad (258)$$

> *SumOverRepeatedIndices*(eq_1)

$$\frac{2 (R_t^2 R_r + 2 R_t R_{r, t} R - 2 E_r R - 2 R_r E)}{R_r R^2} = 16 \pi (\rho_M + \rho_\Lambda) \quad (259)$$

> *isolate*((259), $\rho_M(t, r) + \rho_\Lambda$)

$$\rho_M + \rho_\Lambda = \frac{R_t^2 R_r + 2 R_t R_{r, t} R - 2 E_r R - 2 R_r E}{8 R_r R^2 \pi} \quad (260)$$

> *simplify*((260), {(255)}, { $E(r)$ })

$$\rho_M + \rho_\Lambda = \frac{M_r}{4 R^2 R_r \pi} \quad (261)$$

The second equation, eq_2 , is identically satisfied

> eq_2

$$\begin{aligned}
& \mathcal{D}_\beta(\mathbf{K}_\mu^\beta) - \boldsymbol{\gamma}_\mu^\tau \left(-\frac{1}{R_r R} (2 (R_{r, r} \partial_\tau(r) + R_{r, t} \partial_\tau(t)) R_t + 2 R_r (R_{r, t} \partial_\tau(r) \right. \\
& \left. + R_{t, t} \partial_\tau(t)) + \partial_\tau(R) R_{r, t} + R (R_{r, r, t} \partial_\tau(r) + R_{r, t, t} \partial_\tau(t)) \right) \\
& \left. + \frac{(2 R_r R_t + R R_{r, t}) (R_{r, r} \partial_\tau(r) + R_{r, t} \partial_\tau(t))}{R_r^2 R} + \frac{(2 R_r R_t + R R_{r, t}) \partial_\tau(R)}{R_r R^2} \right)
\end{aligned} \quad (262)$$

$$= -8 \pi \boldsymbol{\gamma}_\mu^\beta \mathbf{n}^\tau T_{\beta, \tau}$$

> *TensorArray(eq₂, simplifier = simplify)*

$$\left[0=0 \quad 0=0 \quad 0=0 \quad 0=0 \right] \quad (263)$$

The fourth equation, *eq₄*, is also identically satisfied (basically, this is the definition of the [ExtrinsicCurvature](#))

> *eq₄*

$$\begin{aligned} t^\tau \mathcal{D}_\tau(\boldsymbol{\gamma}_{\mu, \nu}) + \boldsymbol{\gamma}_{\nu, \tau} \mathcal{D}_\mu(t^\tau) + \boldsymbol{\gamma}_{\mu, \tau} \mathcal{D}_\nu(t^\tau) = -2 \mathbf{K}_{\mu, \nu} + \boldsymbol{\beta}^\tau \mathcal{D}_\tau(\boldsymbol{\gamma}_{\mu, \nu}) + \boldsymbol{\gamma}_{\nu, \tau} \mathcal{D}_\mu(\boldsymbol{\beta}^\tau) \\ + \boldsymbol{\gamma}_{\mu, \tau} \mathcal{D}_\nu(\boldsymbol{\beta}^\tau) \end{aligned} \quad (264)$$

> *TensorArray(eq₂, simplifier = simplify)*

$$\left[0=0 \quad 0=0 \quad 0=0 \quad 0=0 \right] \quad (265)$$

So it is in *eq₃* where the evolution of the gravitational field is encoded, in terms of the functions

$\{\rho_M E(r), R(t, r)\}$ and their derivatives

> *eq₃*

$$t^\tau \mathcal{D}_\tau(\mathbf{K}_{\mu, \nu}) + \mathbf{K}_{\nu, \tau} \mathcal{D}_\mu(t^\tau) + \mathbf{K}_{\mu, \tau} \mathcal{D}_\nu(t^\tau) = - \frac{(2 R_r R_t + R R_{r, t}) \mathbf{K}_{\mu, \nu}}{R_r R} \quad (266)$$

$$\begin{aligned} - 2 \mathbf{K}_{\mu, \tau} \mathbf{K}_{\nu}^\tau + R_{\mu, \nu} - 8 \pi \left(\boldsymbol{\gamma}_\mu^\kappa \boldsymbol{\gamma}_\nu^\lambda T_{\kappa, \lambda} \right. \\ \left. - \frac{\boldsymbol{\gamma}_{\mu, \nu} \left(\boldsymbol{\gamma}_\kappa^\lambda \boldsymbol{\gamma}^{\kappa, \sigma} T_{\lambda, \sigma} - T^{\alpha, \beta} n_\alpha n_\beta \right)}{2} \right) + \boldsymbol{\beta}^\tau \mathcal{D}_\tau(\mathbf{K}_{\mu, \nu}) + \mathbf{K}_{\nu, \tau} \mathcal{D}_\mu(\boldsymbol{\beta}^\tau) \\ + \mathbf{K}_{\mu, \tau} \mathcal{D}_\nu(\boldsymbol{\beta}^\tau) \end{aligned}$$

> *EQ3 := TensorArray(eq₃, simplifier = simplify)*

$$EQ3 := \left[\left[\frac{-R_r R_{r, t, t} - R_{r, t}^2}{1 + 2 E} \right] \right] \quad (267)$$

$$= \frac{-R_{r, t}^2 R + 2 R_r R_{r, t} R_t - 4 R_r \left(\pi R (\rho_M + 2 \rho_\Lambda) R_r + \frac{E_r}{2} \right)}{R (1 + 2 E)}, 0=0, 0=0, 0$$

$$= 0 \left. \right]$$

$$\left[0=0, -R_t^2 - R R_{t, t} = \frac{R_t R_{r, t} R + (-4 \pi (\rho_M + 2 \rho_\Lambda) R^2 - 2 E) R_r - E_r R}{R_r}, 0 \right.$$

$$= 0, 0 = 0 \Big],$$

$$\left[0 = 0, 0 = 0, \left(-R_t^2 - R R_{t,t} \right) \sin(\theta)^2 = \right. \\ \left. - \frac{\sin(\theta)^2 \left(-R_t R_{r,t} R + \left(4 \pi \left(\rho_M + 2 \rho_\Lambda \right) R^2 + 2 E \right) R_r + E_r R \right)}{R_r}, 0 = 0 \right],$$

$$\left[\left[0 = 0, 0 = 0, 0 = 0, 0 = 0 \right] \right]$$

To demonstrate that the system of equations $EQ3$ together with the constraint eq_1 is equivalent to the 4D system of equations $EQ4$ it now suffices to show that each of these two systems entirely reduces the other one. For this purpose, convert these arrays of equations to sets of equations

> $EQ4 := \text{convert}(EQ4, \text{setofequations})$

$$EQ4 := \left\{ 0 = 0, -\rho_\Lambda R^2 = - \frac{R \left(R_t R_{r,t} + R_{r,t,t} R + R_{t,t} R_r - E_r \right)}{8 R_r \pi}, - \frac{\rho_\Lambda R_r^2}{1 + 2 E} = \right. \quad (268) \\ \left. - \frac{R_r^2 \left(R_t^2 + 2 R R_{t,t} - 2 E \right)}{8 R^2 \left(1 + 2 E \right) \pi}, -\rho_\Lambda R^2 \sin(\theta)^2 = \right. \\ \left. - \frac{R \sin(\theta)^2 \left(R_t R_{r,t} + R_{r,t,t} R + R_{t,t} R_r - E_r \right)}{8 R_r \pi}, \rho_M + \rho_\Lambda \right\} \\ = \left\{ \frac{R_t^2 R_r + 2 R_t R_{r,t} R - 2 E_r R - 2 R_r E}{8 R_r R^2 \pi} \right\}$$

> $EQ3 := \text{convert}(EQ3, \text{setofequations}) \cup \{(260)\}$

$$EQ3 := \left\{ 0 = 0, \left(-R_t^2 - R R_{t,t} \right) \sin(\theta)^2 = \right. \quad (269) \\ \left. - \frac{\sin(\theta)^2 \left(-R_t R_{r,t} R + \left(4 \pi \left(\rho_M + 2 \rho_\Lambda \right) R^2 + 2 E \right) R_r + E_r R \right)}{R_r}, \right. \\ \left. \frac{-R_r R_{r,t,t} - R_{r,t}^2}{1 + 2 E} \right\}$$

$$\begin{aligned}
&= \frac{-R_{r,t}^2 R + 2 R_r R_{r,t} R_t - 4 R_r \left(\pi R (\rho_M + 2 \rho_\Lambda) R_r + \frac{E_r}{2} \right)}{R (1 + 2 E)}, -R_t^2 - R R_{t,t} \\
&= \frac{R_t R_{r,t} R + (-4 \pi (\rho_M + 2 \rho_\Lambda) R^2 - 2 E) R_r - E_r R}{R_r}, \rho_M + \rho_\Lambda \\
&= \left. \frac{R_t^2 R_r + 2 R_t R_{r,t} R - 2 E_r R - 2 R_r E}{8 R_r R^2 \pi} \right\}
\end{aligned}$$

The differential reductions can now be performed using [PDEtools:-ReducedForm](#)

> [PDEtools:-ReducedForm](#)(EQ4, EQ3)
[0, 0, 0, 0, 0] &where [] (270)

The reduction the other way around

> [PDEtools:-ReducedForm](#)(EQ3, EQ4)
[0, 0, 0, 0, 0] &where [] (271)

>

▼ Tetrads and Weyl scalars in canonical form

Generally speaking a canonical form is obtained using transformations that leave invariant the tetrad metric in a tetrad system of references, so that the Weyl scalars are fixed as much as possible (conventionally, either equal to 0 or to 1).

Bringing a tetrad in canonical form is a relevant step in the tackling of the equivalence problem between two spacetime metrics.

The implementation is as in "[General Relativity, an Einstein century survey](#)", edited by S.W. Hawking (Cambridge) and W. Israel (U. Alberta, Canada), specifically Chapter 7 written by S. Chandrasekhar, page 388:

	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Residual invariance
Petrov type I	0	$\neq 0$	$\neq 0$	1	0	none
Petrov type II	0	0	$\neq 0$	1	0	none
Petrov type III	0	0	0	1	0	none
Petrov type D	0	0	$\neq 0$	0	0	Ψ_2 remains invariant

						under rotations of Class III
Petrov type N	0	0	0	0	1	Ψ_4 remains invariant under rotations of Class II

The transformations (rotations of the tetrad system of references) used are of Class I, II and III as defined in Chandrasekar's chapter - equations (7.79) in page 384, (7.83) and (7.84) in page 385.

Transformations of Class I can be performed with the command *Physics:-Tetrads:-TransformTetrad* using the optional argument *nullrotationwithfixedl_*, of Class II using *nullrotationwithfixedn_* and of Class III by calling *TransformTetrad(spatialrotationsm_mb_plan, boostsn_l_plane)*, so with the two optional arguments simultaneously.

The determination of appropriate transformation parameters to be used in these rotations, as well as the sequence of transformations happens all automatically by using the optional argument, *canonicalform* of *TransformTetrad*.

> restart;

with (Physics) :

with (Tetrads);

Setting lowercaselatin_ah letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism (see

?Physics,tetrads), $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$

[IsTetrad, NullTetrad, OrthonormalTetrad, PetrovType, SegreType, TransformTetrad, e_ (272)

eta_, gamma_, l_, lambda_, m_, mb_, n_]

▼ Petrov type I

The numbers below used to enter the metric always refer to the equation number in the "[Exact solutions to Einstein's field equations](#)" textbook

> g_ [[12, 21, 1]]

Systems of spacetime Coordinates are: $\{X = (t, x, y, \phi)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (t, x, y, \phi)\}$

The McLenaghan, Tariq (1975), Tupper (1976) metric in coordinates $[t, x, y, \phi]$

Parameters: $[a, k, \kappa 0]$

Comments: k parametrizes the most general electromagnetic invariant with respect to the last 3 Killing vectors

Resetting the signature of spacetime from "- - - +" to "- + + +" in order to match the

signature in the database of metrics:

$$g_{\mu, \nu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a^2}{x^2} & 0 & 0 \\ 0 & 0 & \frac{a^2}{x^2} & 0 \\ 2y & 0 & 0 & x^2 - 4y^2 \end{bmatrix} \quad (273)$$

The default tetrad computed by the Physics package routines

> `e_[]`

$$e_{a, \mu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a}{|x|} & 0 & 0 \\ 0 & 0 & \frac{a}{|x|} & 0 \\ 0 & 0 & 0 & |x| \end{bmatrix} \quad (274)$$

The corresponding Weyl scalars

> `Weyl[scalars]`

$$\Psi_0 = \frac{I|x|}{a^2 x}, \Psi_1 = 0, \Psi_2 = -\frac{1}{a^2}, \Psi_3 = 0, \Psi_4 = \frac{I|x|}{a^2 x} \quad (275)$$

... there is abs around. Let's assume everything is positive to simplify the presentation of formulas

> `Assume(x > 0, y > 0, a > 0)`

$$\{a::(0, \infty)\}, \{x::(0, \infty)\}, \{y::(0, \infty)\} \quad (276)$$

The scalars are now simpler, although still not in "canonical form" because $\Psi_4 \neq 0$ and $\Psi_3 \neq 1$.

> `Weyl[scalars]`

$$\Psi_0 = \frac{I}{a^2}, \Psi_1 = 0, \Psi_2 = -\frac{1}{a^2}, \Psi_3 = 0, \Psi_4 = \frac{I}{a^2} \quad (277)$$

The Petrov type

> `PetrovType()`

$$\text{"I"} \quad (278)$$

In this case the Weyl scalars are in canonical form when $\Psi_0 = 0$, $\Psi_4 = 0$ and $\Psi_3 = 1$.

> `TransformTetrad(canonicalform)`

$$\left[\left[-\frac{\sqrt{\sqrt{2}\sqrt{5}-3}(2\sqrt{2}+\sqrt{5})}{2a^2}, \frac{\sqrt{\sqrt{2}\sqrt{5}-3}(\sqrt{2}+\sqrt{5})}{2ax}, \frac{1}{2ax}, \right. \right. \quad (279)$$

$$\begin{aligned}
& \frac{1}{6a^2} \left(\sqrt{\sqrt{2}\sqrt{5}-3} (2\sqrt{2}+\sqrt{5}) \left(x\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2} \right. \right. \\
& \left. \left. + \sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}x+6y \right) \right), \\
& \left[-\frac{\sqrt{\sqrt{2}\sqrt{5}-3}(5\sqrt{2}+4\sqrt{5})a^2}{10}, \right. \\
& \left. -\frac{\sqrt{\sqrt{2}\sqrt{5}-3}a^3(5\sqrt{2}+2\sqrt{5})}{10x}, -\frac{\sqrt{2}\sqrt{5}a^3}{10x}, \right. \\
& \left. \frac{1}{30} \left(\sqrt{\sqrt{2}\sqrt{5}-3}a^2(5\sqrt{2}+4\sqrt{5}) \left(x\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2} \right. \right. \right. \\
& \left. \left. + \sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}x+6y \right) \right) \right], \\
& \left[-\frac{I\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}}{4} - \frac{I\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}}{5} + \frac{\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}}{4} \right. \\
& \left. + \frac{\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}}{10}, \right. \\
& \left. -\frac{\sqrt{\sqrt{2}\sqrt{5}-3}(5I\sqrt{2}+2I\sqrt{5}+5\sqrt{2}+4\sqrt{5})a}{20x}, \right. \\
& \left. -\frac{(-10I+I\sqrt{2}\sqrt{5}-3\sqrt{2}\sqrt{5})a}{20x}, \frac{1}{60} \left((10I+I\sqrt{2}\sqrt{5} \right. \right. \\
& \left. \left. -3\sqrt{2}\sqrt{5}) \left(2\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}y+2y\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}+3x \right) \right) \right], \\
& \left[\frac{I\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}}{4} + \frac{\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}}{4} + \frac{I\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}}{5} \right. \\
& \left. + \frac{\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}}{10}, \right. \\
& \left. \frac{\sqrt{\sqrt{2}\sqrt{5}-3}(5I\sqrt{2}+2I\sqrt{5}-5\sqrt{2}-4\sqrt{5})a}{20x}, \right. \\
& \left. \frac{(-10I+I\sqrt{2}\sqrt{5}+3\sqrt{2}\sqrt{5})a}{20x}, -\frac{1}{60} \left((10I+I\sqrt{2}\sqrt{5} \right. \right. \\
& \left. \left. +3\sqrt{2}\sqrt{5}) \left(2\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{5}y+2y\sqrt{\sqrt{2}\sqrt{5}-3}\sqrt{2}+3x \right) \right) \right]
\end{aligned}$$

Despite the fact that the result is a much more complicated tetrad, this is an amazing result in that the resulting Weyl scalars are all fixed (see below). Let's first verify that this is indeed a tetrad, and that now the Weyl scalars are in canonical form

> *IsTetrad*((279))

Type of tetrad: null

true

(280)

Set (279) to be the tetrad in use and recompute the Weyl scalars

> *Setup*(tetrad = (279)) :

Inded we now have $\Psi_0 = 0$, $\Psi_4 = 0$ and $\Psi_3 = 1$

> *simplify*([*Weyl*[*scalars*]])

$$\left[\Psi_0 = 0, \Psi_1 = \frac{-\frac{1}{2} - \frac{3I}{2}}{a^4}, \Psi_2 = \frac{-1 + I}{a^2}, \Psi_3 = 1, \Psi_4 = 0 \right] \quad (281)$$

So Weyl scalars computed after setting the canonical tetrad (279) to be the tetrad in use are in canonical form. Great! NOTE: computing the canonicalWeyl scalars is not really the difficult part, and within the code, these scalars (281) are computed before arriving at the tetrad (279). What is really difficult (from the point of view of computational complexity and simplifications) is to compute the actual canonical form of the tetrad (279).

>

▼ Petrov type II

Consider this other solution to Einstein's equation (again, the numbers in $g_{[[24,37,7]]}$ always refer to the equation number in the "[Exact solutions to Einstein's field equations](#)" textbook)

> $g_{[[24, 37, 7]]}$

Systems of spacetime Coordinates are: {X = (u, v, x, y)}

Default differentiation variables for d_, D_ and dAlembertian are: {X = (u, v, x, y)}

The Stephani metric in coordinates [u, v, x, y]

Parameters: [f(x), a, Psi1(u, x, y)]

Comments: Case 6 from Table 24.1: $\text{Psi1}(u, x, y): \text{diff}(\text{Psi1}(u, x, y), x, x) + \text{diff}(\text{Psi1}(u, x, y), y, y) = 0$, $\text{diff}(x * \text{diff}(_M(u, x, y), x), x) + x * \text{diff}(_M(u, x, y), y, y) = _kappa0 * (\text{diff}(_Psi(u, x, y), x)^2 + \text{diff}(_Psi(u, x, y), y)^2)$

$$g_{\mu, \nu} = \begin{bmatrix} -2x(f(x) + ya) & -x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{x}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{x}} \end{bmatrix} \quad (282)$$

Check the Petrov type

> *PetrovType*()

"II"

(283)

The starting tetrad

> $e_{[]}$

$$e_{a, \mu} = \begin{pmatrix} -\sqrt{x} \sqrt{f(x) + ya} & 0 & 0 & 0 \\ -\sqrt{x} \sqrt{f(x) + ya} - \frac{\sqrt{x}}{\sqrt{f(x) + ya}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2x^{1/4}} & \frac{1}{2} \frac{\sqrt{2}}{x^{1/4}} \\ 0 & 0 & \frac{\sqrt{2}}{2x^{1/4}} & -\frac{1}{2} \frac{\sqrt{2}}{x^{1/4}} \end{pmatrix} \quad (284)$$

results in Weyl scalars not in canonical form:

> *Weyl[scalars]*

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = \frac{1}{8x^{3/2}}, \Psi_3 = 0, \Psi_4 = \quad (285)$$

$$\frac{3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right)}{\sqrt{x} (4ya + 4f(x))}$$

For Petrov type "II", the canonical form is as for type "I" but in addition $\Psi_1 = 0$. Again let's assume positive, not necessary, but to get simpler formulas around

> *Assume*($f(x) > 0, x > 0, y > 0, a > 0$)

$$\{a::(0, \infty)\}, \{x::(0, \infty)\}, \{-f(x)::[-\infty, 0)\}, \{f(x)::(0, \infty)\}, \{y::(0, \infty)\} \quad (286)$$

Compute now a canonical form for the tetrad, to be used instead of (284)

> *TransformTetrad*(canonicalform)

$$\left[\left[\frac{\sqrt{3} \sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|}}{8\sqrt{x}}, 0, 0, 0 \right], \quad (287)$$

[

$$-\left(8\sqrt{3} x^{3/2} \left(x \left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right| + 3ya \right)$$

$$+ 3f(x)) \Big) \Big) / \left(9 \sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \right),$$

$$- \frac{8\sqrt{3} x^{3/2}}{3 \sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|}}, - \left(4x^{5/4} \sqrt{2} \left(3Ia \right.$$

$$\left. - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) + \left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right| \right)$$

$$f(x) \Big) \Big) \Big) \Big) /$$

$$\left(3 \sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \right)$$

$$\sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|}, \left(4\sqrt{2} x^{5/4} \left(2I \left(\frac{d^2}{dx^2} f(x) \right) x + 3I \left(\frac{d}{dx} f(x) \right) + I \left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right| + 3a \right) \right) \Big) \Big) /$$

$$\left(3 \sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \right)$$

$$\sqrt{\left| 3Ia - 2x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \Big) \Big) ,$$

$$\left[\begin{array}{l}
-\frac{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{3} x}{3}, 0, \\
-\frac{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{2}}{2 \sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} x^{1/4}}, \\
-\frac{\frac{1}{2} \sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{2}}{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} x^{1/4}} \right], \\
\left[\begin{array}{l}
-\frac{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{3} x}{3}, 0, \\
-\frac{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{2}}{2 \sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} x^{1/4}}, \\
\frac{\frac{1}{2} \sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} \sqrt{2}}{\sqrt{\left| 3 I a - 2 x \left(\frac{d^2}{dx^2} f(x) \right) - 3 \left(\frac{d}{dx} f(x) \right) \right|} x^{1/4}} \right]
\end{array} \right.$$

Set this tetrad and check the Weyl scalars again

> Setup(tetrad = (287)) :

> Weyl[scalars]

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = \frac{1}{8 x^3 / 2}, \Psi_3 = 1, \Psi_4 = 0 \quad (288)$$

This result (288) is fantastic. Compare these Weyl scalars with the ones (285) before transforming the tetrad.

>

▼ Petrov type III

> g_([[12, 35, 1]])

Systems of spacetime Coordinates are: $\{X = (u, x, y, z)\}$

Default differentiation variables for d , D and dA lembertian are: $\{X = (u, x, y, z)\}$

The Kaigorodov (1962), Cahen (1964), Siklos (1981), Ozsvath (1987) metric in coordinates $[u, x, y, z]$

Parameters: $[\Lambda]$

Warning, for the signature $(- + + +)$, that is with the timelike component in position

1, the spacetime metric indicated has $g_{0,0} = g_{1,1} = 0$, and so the corresponding system of reference cannot be realized with real bodies (e.g. you cannot define proper time nor synchronize clocks in any infinitesimal region of space). Note as well that the corresponding 3-dimensional space metric γ is singular.

$$g_{\mu, \nu} = \begin{bmatrix} 0 & e^{-2z} & 0 & 0 \\ e^{-2z} & e^{4z} & 2e^z & 0 \\ 0 & 2e^z & 2e^{-2z} & 0 \\ 0 & 0 & 0 & \frac{3}{|\Lambda|} \end{bmatrix} \quad (289)$$

> Assume($z > 0$, $\Lambda > 0$)

$$\{\Lambda :: (0, \infty]\}, \{z :: (0, \infty]\} \quad (290)$$

The Petrov type and the original tetrad

> PetrovType()

$$\text{"III"} \quad (291)$$

> $e_{[}$

$$e_{a, \mu} = \begin{bmatrix} -\frac{1}{2} e^{-4z} (\sqrt{2} - 2) & -\frac{1}{2} \sqrt{2} e^{2z} & -e^{-z} (\sqrt{2} - 1) & 0 \\ -\frac{1}{2} e^{-4z} (2 + \sqrt{2}) & -\frac{1}{2} \sqrt{2} e^{2z} & -e^{-z} (1 + \sqrt{2}) & 0 \\ \frac{1}{2} \sqrt{2} e^{-4z} & 0 & 0 & \frac{\sqrt{2} \sqrt{3}}{2\sqrt{\Lambda}} \\ -\frac{1}{2} \sqrt{2} e^{-4z} & 0 & 0 & \frac{\sqrt{2} \sqrt{3}}{2\sqrt{\Lambda}} \end{bmatrix} \quad (292)$$

This tetrad results in the following scalars

> Weyl[scalars]

$$\Psi_0 = \frac{11\Lambda}{4} - 2\Lambda\sqrt{2}, \Psi_1 = \frac{3\Lambda}{4} - \frac{\Lambda\sqrt{2}}{2}, \Psi_2 = \frac{\Lambda}{4}, \Psi_3 = -\frac{3\Lambda}{4} - \frac{\Lambda\sqrt{2}}{2}, \Psi_4 \quad (293)$$

$$= \frac{11 \Lambda}{4} + 2 \Lambda \sqrt{2}$$

that are not in canonical form, which for Petrov type III is as in Petrov type II but in addition we should have $\Psi_2 = 0$.

Compute now a canonical form for the tetrad

> *TransformTetrad (canonicalform)*

$$\begin{bmatrix} 0 & -\frac{1}{2} \sqrt{2} \Lambda e^{2z} & -\frac{1}{2} \sqrt{2} e^{-z} \Lambda & \frac{\sqrt{3} \sqrt{\Lambda}}{2} \\ \frac{1}{2} \sqrt{2} e^{-4z} & -\frac{13}{8} \frac{1}{\Lambda} \sqrt{2} e^{2z} & -\frac{9}{8} \frac{1}{\Lambda} \sqrt{2} e^{-z} & -\frac{7 \sqrt{3}}{8 \Lambda^{3/2}} \\ 0 & \frac{3}{4} \frac{1}{\Lambda} \sqrt{2} e^{2z} & \frac{1}{4} \sqrt{2} e^{-z} & -\frac{\sqrt{3}}{4 \sqrt{\Lambda}} \\ -1 e^{-4z} \sqrt{2} & \frac{3}{4} \frac{1}{\Lambda} \sqrt{2} e^{2z} & \frac{1}{4} \sqrt{2} e^{-z} & -\frac{\sqrt{3}}{4 \sqrt{\Lambda}} \end{bmatrix} \quad (294)$$

Set this one to be the tetrad in use and recompute the Weyl scalars

> *Setup (tetrad = (294)) :*

> *Weyl[scalars]*

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = 0, \Psi_3 = 1, \Psi_4 = 0 \quad (295)$$

>

▼ Petrov type N

> *g_[[12, 6, 1]]*

Systems of spacetime Coordinates are: {X = (u, v, y, z)}

Default differentiation variables for d_, D_ and dAlembertian are: {X = (u, v, y, z)}

The Defrise (1969) metric in coordinates [u, v, y, z]

Parameters: [Λ, κ0]

Comments: Lambda < 0 required for a pure radiation solution

Warning, for the signature (- + + +), that is with the timelike component in position

1, the spacetime metric indicated has $g_{0,0} = g_{1,1} = 0$, and so the corresponding system of reference cannot be realized with real bodies (e.g. you cannot define proper time nor synchronize clocks in any infinitesimal region of space). Note as well that the corresponding 3-dimensional space metric γ is singular.

$$g_{\mu, \nu} = \begin{bmatrix} 0 & -\frac{3}{2y^2 \Lambda} & 0 & 0 \\ -\frac{3}{2y^2 \Lambda} & \frac{3}{\Lambda y^4} & 0 & 0 \\ 0 & 0 & \frac{3}{y^2 \Lambda} & 0 \\ 0 & 0 & 0 & \frac{3}{y^2 \Lambda} \end{bmatrix} \quad (296)$$

> *Assume*($y > 0$, $\Lambda > 0$)
 $\{\Lambda::(0, \infty]\}, \{y::(0, \infty]\}$ (297)

> *PetrovType*()
 "N" (298)

The original tetrad and related Weyl scalars are not in canonical form:

> e_{μ}^a]

$$e_{a, \mu} = \begin{bmatrix} \frac{\frac{1}{4} \sqrt{2} \sqrt{3}}{\sqrt{\Lambda}} & -\frac{\frac{1}{2} \sqrt{2} \sqrt{3}}{\sqrt{\Lambda} y^2} & \frac{\sqrt{2} \sqrt{3}}{2y \sqrt{\Lambda}} & 0 \\ \frac{\frac{1}{4} \sqrt{2} \sqrt{3}}{\sqrt{\Lambda}} & -\frac{\frac{1}{2} \sqrt{2} \sqrt{3}}{\sqrt{\Lambda} y^2} & -\frac{\sqrt{2} \sqrt{3}}{2y \sqrt{\Lambda}} & 0 \\ -\frac{\sqrt{2} \sqrt{3}}{4 \sqrt{\Lambda}} & 0 & 0 & \frac{\sqrt{2} \sqrt{3}}{2y \sqrt{\Lambda}} \\ \frac{\sqrt{2} \sqrt{3}}{4 \sqrt{\Lambda}} & 0 & 0 & \frac{\sqrt{2} \sqrt{3}}{2y \sqrt{\Lambda}} \end{bmatrix} \quad (299)$$

> *Weyl*[*scalars*]

$$\Psi_0 = -\frac{\Lambda}{4}, \Psi_1 = \frac{1}{4} \Lambda, \Psi_2 = \frac{\Lambda}{4}, \Psi_3 = -\frac{1}{4} \Lambda, \Psi_4 = -\frac{\Lambda}{4} \quad (300)$$

For Petrov type "N", the canonical form has $\Psi_4 \neq 0$ and all the other $\Psi_n = 0$.

Compute a canonical form, set it to be the tetrad in use and recompute the Weyl scalars

> *TransformTetrad*(*canonicalform*)

$$\begin{bmatrix}
 0 & \frac{-\frac{1}{2}\sqrt{2}\sqrt{3}}{y^2} & 0 & 0 \\
 \frac{\frac{1}{2}\sqrt{2}\sqrt{3}}{\Lambda} & \frac{-\sqrt{2}\sqrt{3}}{\Lambda y^2} & -\frac{\sqrt{2}\sqrt{3}}{\Lambda y} & 0 \\
 0 & \frac{-\frac{1}{2}\sqrt{2}\sqrt{3}}{\sqrt{\Lambda}y^2} & -\frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & \frac{\frac{1}{2}\sqrt{2}\sqrt{3}}{y\sqrt{\Lambda}} \\
 0 & \frac{-\frac{1}{2}\sqrt{2}\sqrt{3}}{\sqrt{\Lambda}y^2} & -\frac{\sqrt{2}\sqrt{3}}{2y\sqrt{\Lambda}} & -\frac{\frac{1}{2}\sqrt{2}\sqrt{3}}{y\sqrt{\Lambda}}
 \end{bmatrix} \quad (301)$$

> Setup(tetrad = (301) :

> Weyl[scalars]

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = 0, \Psi_3 = 0, \Psi_4 = 1 \quad (302)$$

>

▼ Petrov type D

> g_[[12, 8, 4]]

Systems of spacetime Coordinates are: $\{X = (t, x, y, z)\}$

Default differentiation variables for $d_$, $D_$ and $dAlembertian$ are: $\{X = (t, x, y, z)\}$

The metric in coordinates $[t, x, y, z]$

Parameters: $[A, B]$

Comments: $k = 0, kprime = 1$

$$g_{\mu, \nu} = \begin{bmatrix} -B^2 \sin(z)^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & A^2 x^2 & 0 \\ 0 & 0 & 0 & B^2 \end{bmatrix} \quad (303)$$

> Assume $\left(A > 0, B > 0, x > 0, 0 \leq z \leq \frac{\text{Pi}}{4}\right)$

$$\{A::(0, \infty]\}, \{B::(0, \infty]\}, \{x::(0, \infty]\}, \left\{z::\left[0, \frac{\pi}{4}\right]\right\} \quad (304)$$

> PetrovType()

$$\text{"D"} \quad (305)$$

The default tetrad and related Weyl scalars are not in canonical form, which for Petrov type "D" is with $\Psi_2 \neq 0$ and all the other $\Psi_n = 0$

> $e_{[\]}$

$$e_{a, \mu} = \begin{bmatrix} \frac{\sqrt{2} B \sin(z)}{2} & \frac{\sqrt{2} A}{2} & 0 & 0 \\ \frac{\sqrt{2} B \sin(z)}{2} & -\frac{\sqrt{2} A}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2} A x}{2} & \frac{1}{2} \sqrt{2} B \\ 0 & 0 & \frac{\sqrt{2} A x}{2} & -\frac{1}{2} \sqrt{2} B \end{bmatrix} \quad (306)$$

> *Weyl[scalars]*

$$\Psi_0 = \frac{1}{4 B^2}, \Psi_1 = 0, \Psi_2 = \frac{1}{12 B^2}, \Psi_3 = 0, \Psi_4 = \frac{1}{4 B^2} \quad (307)$$

Transform the tetrad, set it and recompute the Weyl scalars

> *TransformTetrad(canonicalform)*

$$\begin{bmatrix} \sqrt{2} B \sin(z) & 0 & 0 & B \sqrt{2} \\ \frac{\sqrt{2} B \sin(z)}{4} & 0 & 0 & -\frac{B \sqrt{2}}{4} \\ 0 & -\frac{1}{2} \sqrt{2} A & \frac{\sqrt{2} A x}{2} & 0 \\ 0 & \frac{1}{2} \sqrt{2} A & \frac{\sqrt{2} A x}{2} & 0 \end{bmatrix} \quad (308)$$

> *Setup(tetrad = (308)) :*

> *Weyl[scalars]*

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = -\frac{1}{6 B^2}, \Psi_3 = 0, \Psi_4 = 0 \quad (309)$$

Again the expected canonical form of the Weyl scalars, and $\Psi_2 \neq 0$ remains invariant under transformations of Class III.

>