num211

We are looking for the vector valued solution $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ of the following system of ODEs

$$x' = Mx + b(t) =: f(t, x)$$

where the matrix $M \in \mathbb{R}^{n \times n}$ and the vector valued function $b(t)$ are given. So, M can be constant. Additionally the initial value $x(0) =: x_0 \in \mathbb{R}^n$ is given (as a vector).

One way to approximate the solution at $t = h$ numerically, is to choose an matrix $A = (a_{ij}) \in \mathbb{R}^{s \times s}$, a parameter vector $c = (c_i) \in \mathbb{R}^s$ $(0 \leq c_i \leq 1)$ and a vector $b = (b_i) \in \mathbb{R}^s$. And write down the following system of $s \cdot n$ equations and $s \cdot n$ unknowns:

$$\begin{aligned} x_1 &= x_0 + h(a_{11} f(c_1 h, x_1) + \ldots + a_{1s} f(c_s h, x_s)) \\ &\vdots \\ x_s &= x_0 + h(a_{s1} f(c_1 h, x_1) + \ldots + a_{ss} f(c_s h, x_s)) \end{aligned}$$

where $x_i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,n} \end{pmatrix} \in \mathbb{R}^n, i = 1, \ldots, s$ have to be determined. After this system is solved, an approximation of the solution $x(t)$ at $t = h$ is given by

$$x_0 + h(b_1 f(c_1 h, x_1) + \ldots + b_s f(c_s h, x_s))$$

Of course, this system does not have a solution for arbitrary A, b, c. But the following example should have a solution: Example: $A = \begin{pmatrix} 0 & 0 \\ 1/3 & 1/3 \end{pmatrix}, b = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$

And lets say we want to approximate the solution of $x' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 \\ t \\ 2t \end{pmatrix}$ at some small t for $x_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. (or any other example you like)