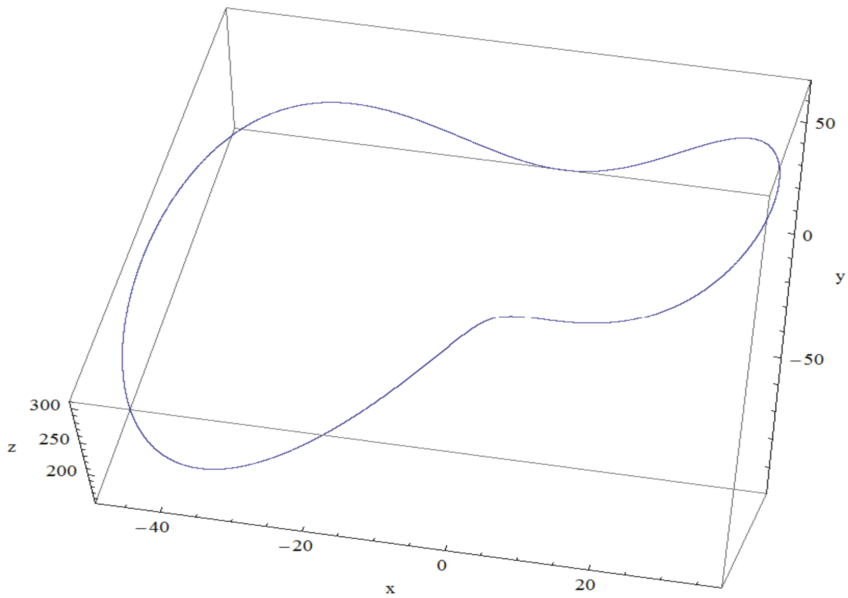
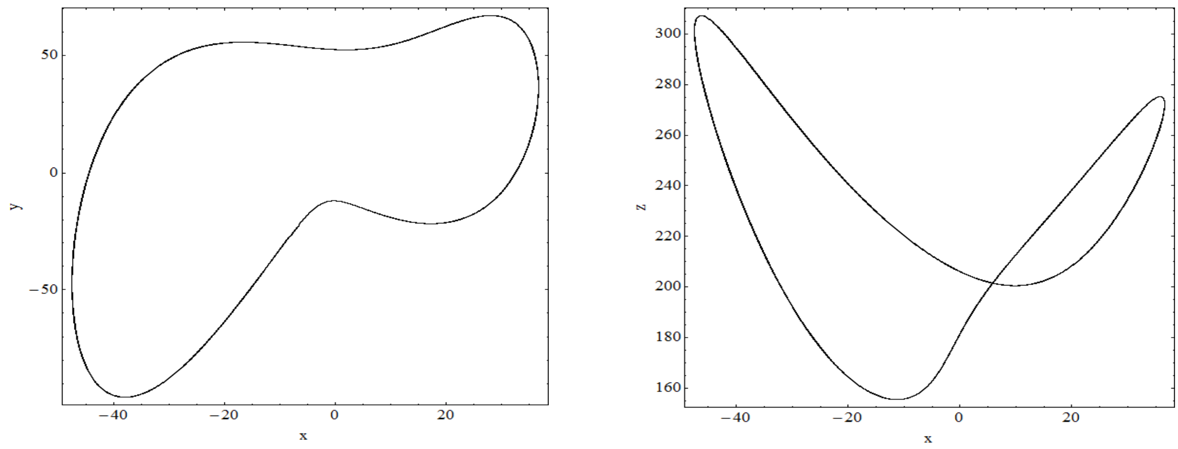
The Lorenz system (6.1) is a strongly dissipative system and hence the Poincare map in this case can be considered as one dimensional [35]. It is further to be noted that we cannot find an analytical expression for the Poincare map in this case but we can numerically find the points where the trajectories cuts the Poincare section and draw our conclusions depending on those points. Below, we have mentioned our technique of finding those points with the help of the phase portraits.

We have initially solved the Lorenz system numerically with the help of classical Runge-Kutta fourth order method with some chosen initial points with the parameter values $\sigma = 10, \beta = 8/3$ and for different values of r which are labelled in the figures and have drawn the trajectories in the phase space. These figures are shown in figures 6.5(a), 6.6(a), 6.7(a), 6.8(a) and 6.9(a). Next, we have projected those phase portraits in the plane $z = 0$. The projected phase portraits are shown in figures 6.5(b), 6.6(b), 6.7(b), 6.8(b) and 6.9(b). We then considered the Poincare section to be the line $x = y$ and found out where the projected trajectories cut the above mentioned section when they move from 'right-to-left'. These points are shown in figures 6.5(c), 6.6(c), 6.7(c), 6.8(c) and 6.9(c). In (d), (e) and (f) of figures 6.5, 6.6, 6.7, 6.8 and 6.9,we have further shown the time series plots of x, y, z coordinates respectively. Depending on the above mentioned figures we can now draw conclusions about the Lorenz system in the following way:

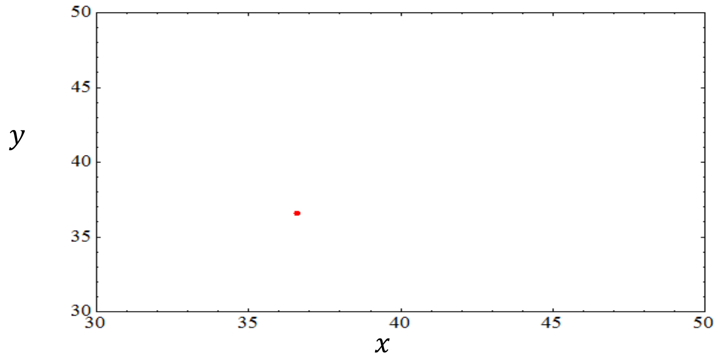


(a)



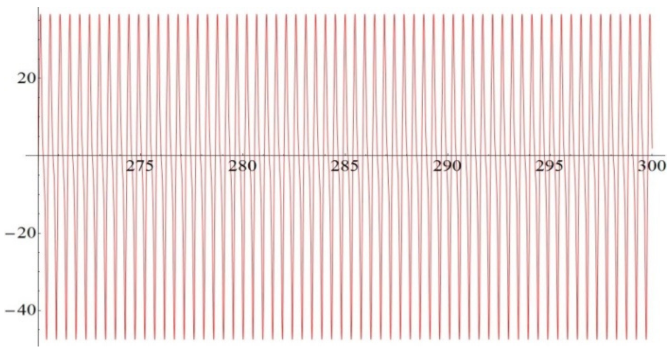
50
0
y
−50
−40
−20
0
20
x
300
280
260
240
z
220
200
180
160
−40
−20
0
20
x

(b)



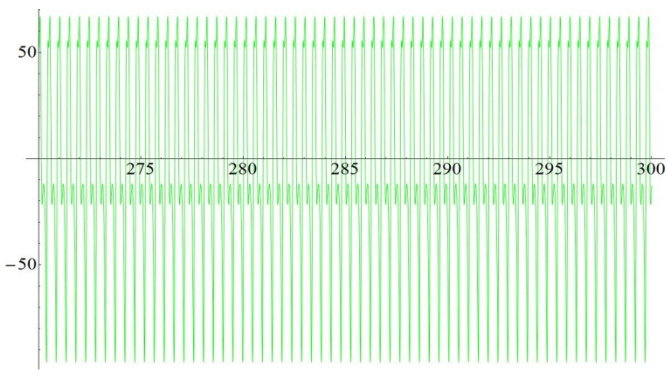
50
45
y
40
35
30
30
35
40
45
50
x

(c)



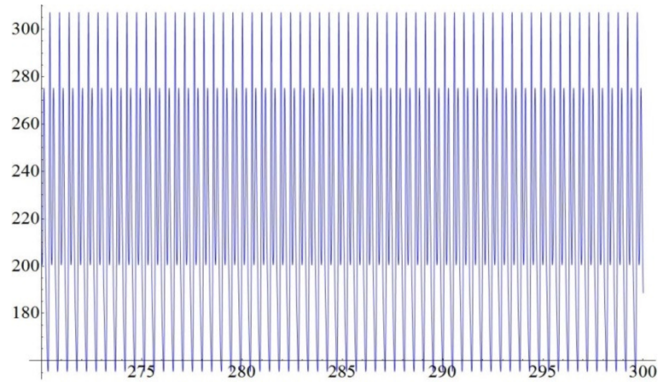
20
275
280
285
290
295
300
−20
−40

(d)



50
275
280
285
290
295
300
−50

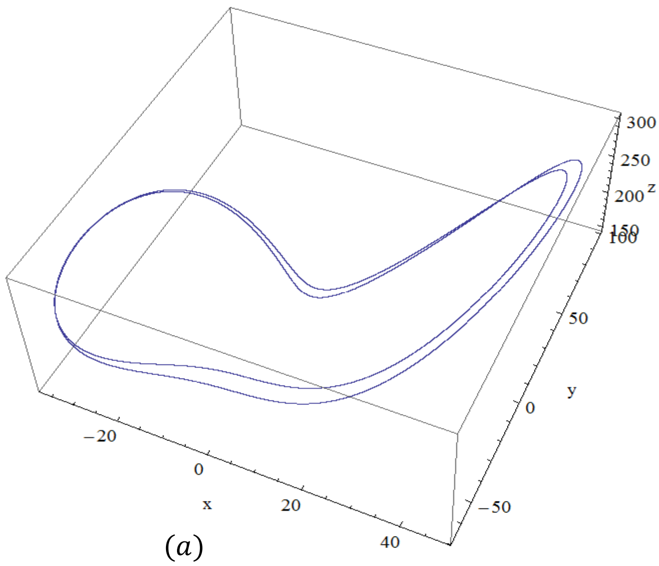
(e)



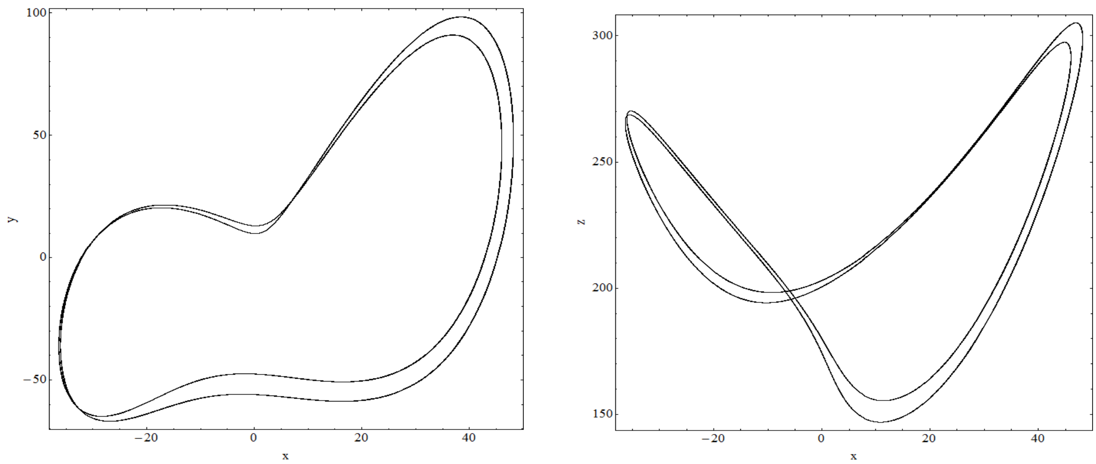
(f)

Figure 6.5:(a)Phase portraits of Lorenzsystem in the phase space when $\sigma = 10, b = \frac{8}{3}, r = 230$ with initial point $x = -8, y = 1, z = 24$ (b). Projection of (a) on the xy-plane and xz-plane (c).Trace of (b) in Poincare section (d).time series plot of x-values (e). time series plot of y-values (f). Time series plot of z-values which shows period one behaviour.

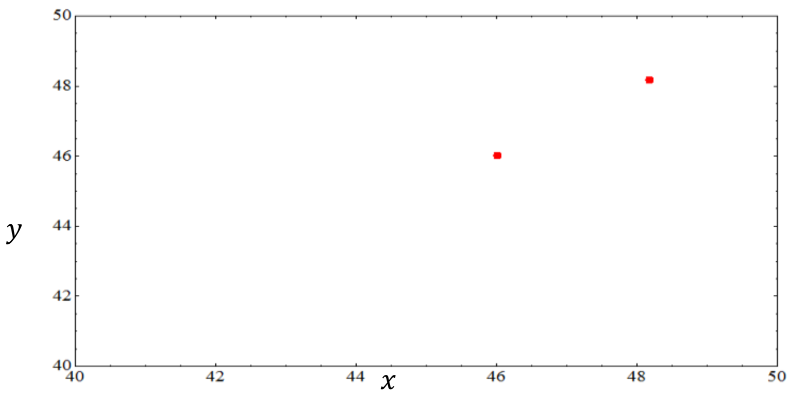
The above figures are drawn for $\sigma = 10,\ b = \frac{8}{3}$ and $r = 230$. In the Poincare section (Fig. 6.5(c)), we get a single point when we vary $r$continuously from ∞ to $229.412 \ldots$ keeping the other two parameters σ and b fixed[139]. This implies that the stable period one behaviour of the limit cycle continues within this parameter range. This fact is also supported by the time series plots which are shown in figures 6.5(d), 6.5(e) and 6.5(f). But as soon as we cross the parameter value $r = 229.412 \ldots$, the situation changes drastically. Instead of getting a single point, this time we get two points in the Poincare section which shows that the periodicity of the limit cycle increases from one to two. The situation is shown in the following Fig. 6.6. Hence, we consider the first period doubling point to be $r = 229.412 \ldots$. At this point, we want to mention that we have drawn our conclusion depending on the behaviour of one variable which we observed in case of our one dimensional Poincare map. Investigation in this direction have already established the fact that if Period doubling occurs for one variable, then it generally occurs for the others as well. Hence, following the behaviour of one variable allows us to determine the overall periodicity and bifurcations for the system [72]. To validate this finding we have drawn the time series plots for *x*, *y* and *z* in all the cases and have seen that they are in conformity with the above mentioned fact.



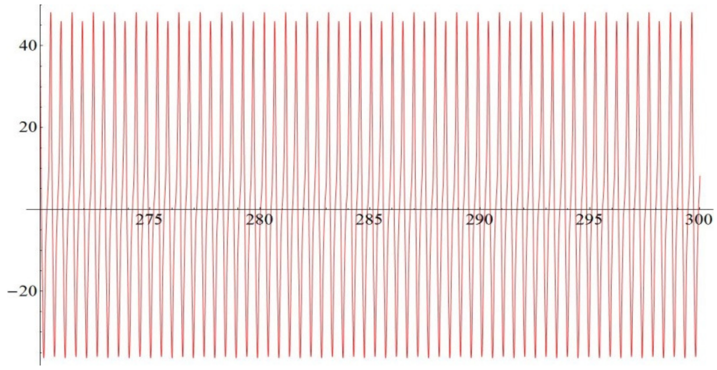
(a)



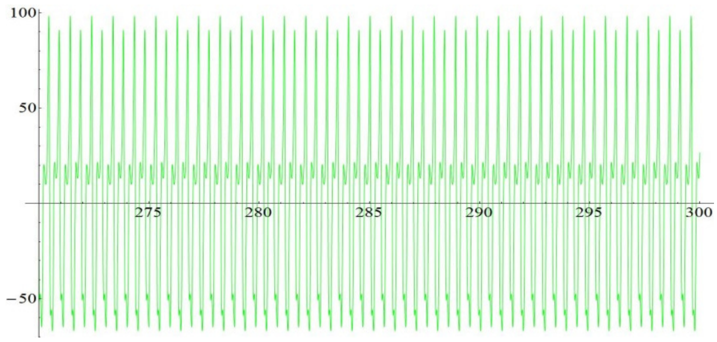
(b)



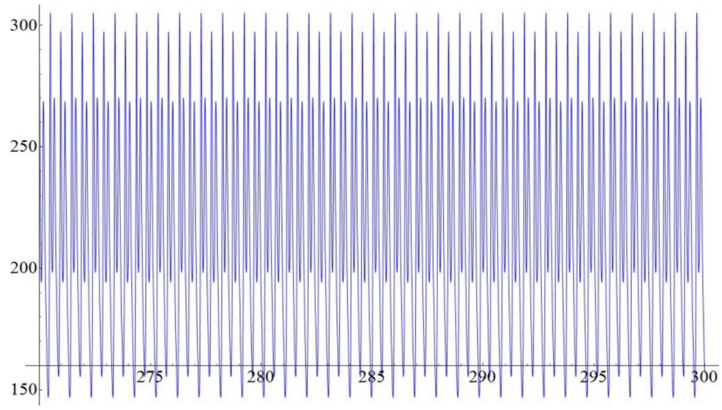
(c)



(d)



(e)



(f)

Figure 6.6: (a) Phase portraits of Lorenz system in the phase space when $\sigma = 10, \beta = \frac{8}{3}, r = 225$ with initial point $x = 0, y = 1, z = 0$ (b). Projection of (a) on the xy-plane and xz-plane (c). Trace of (b) in Poincare section (d).time series plot of x-values (e). time series plot of y-values (f). time series plot of z-values which shows period two behaviour.

We have followed the same technique as elaborated above to find out several period doubling bifurcation points as listed below :

1st bifurcation point $r1 = 229.412$ ...… where the period of the limit cycle changes from 1 to 2

2nd bifurcation point $r2 = 218.2$ ... where the period of the limit cycle changes from 2 to 4

3rd bifurcation point $r3 = 215.9665\ldots$ where the period of the limit cycle changes from 4 to 8

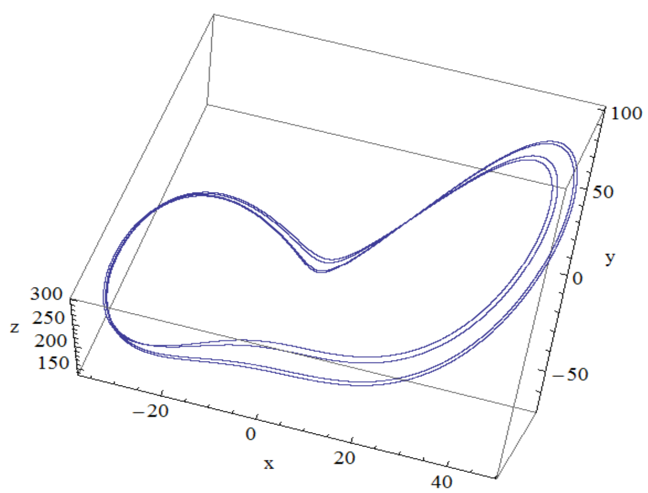
4th bifurcation point $r4 = 215.49231\ldots$ where the period of the limit cycle changes from 8 to 16

5th bifurcation point $r5 = 215.3908\ldots$ where the period of the limit cycle changes from 16 to 32

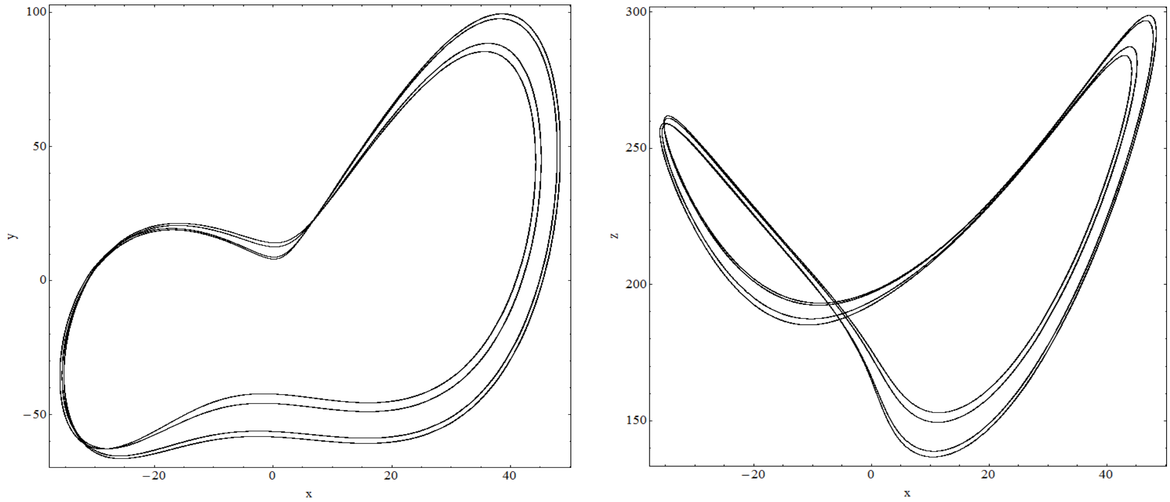
6th bifurcation point $r6 = 215.369057\ldots$ where the period of the limit cycle changes from 32 to 64

7th bifurcation point $r7 = 215.36440296\ldots$ where the period of the limit cycle changes from 64 to 128 and so on.

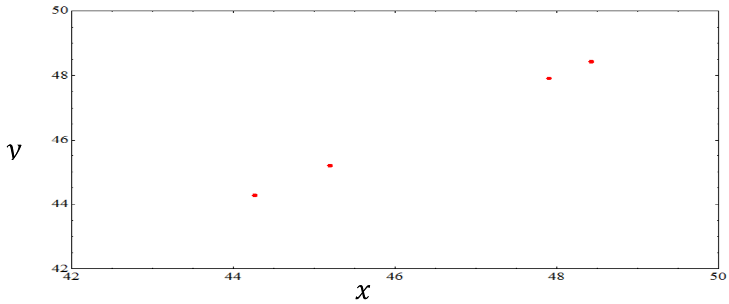
We further want to mention that following the technique we have explained above, it is possible to find more and more bifurcation points but it takes long duration of computer time to detect such a bifurcation point. Moreover, from the above bifurcation points it is seen that the difference between the successive bifurcation points goes on decreasing.



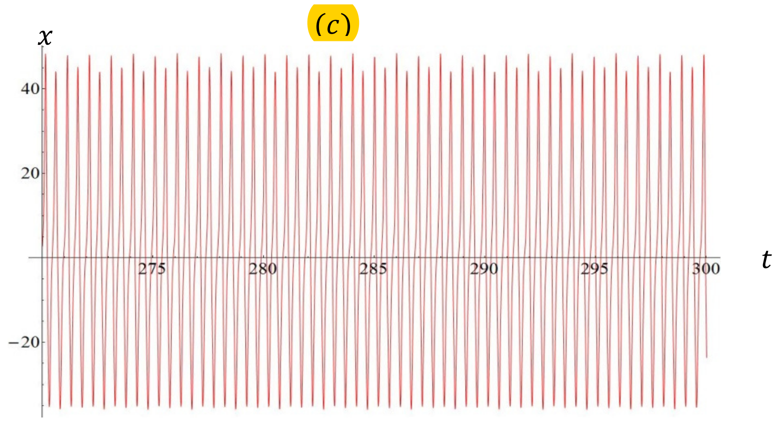
(a)


100
50
0
−50
−20
0
20
40
x
y
300
250
200
150
−20
0
20
40
x
z

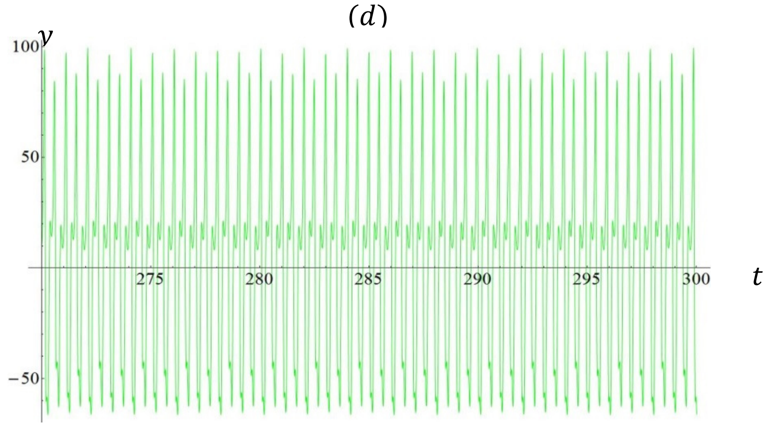
(b)


50
48
46
44
42
42
44
46
48
50
ν
x

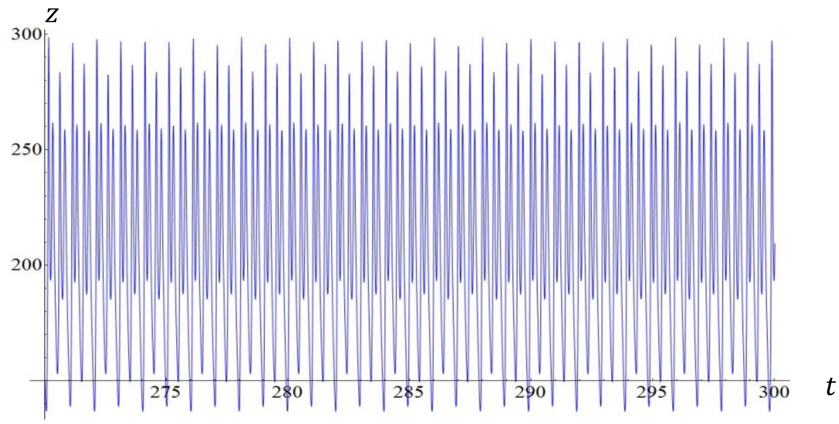
(c)


x
40
20
−20
275
280
285
290
295
300
t

(d)

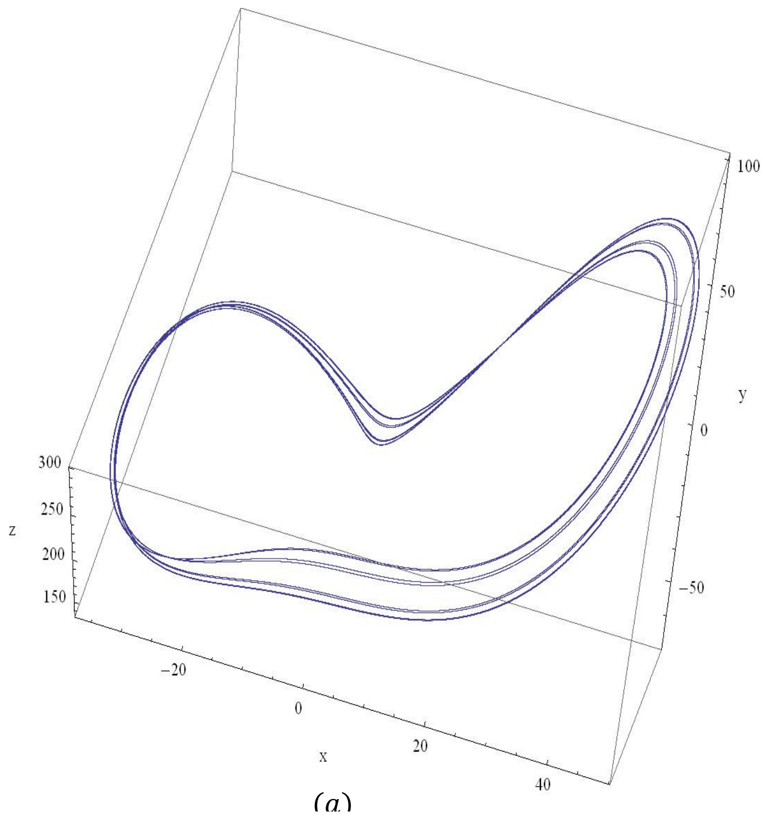

y
100
50
−50
275
280
285
290
295
300
t

z (e)

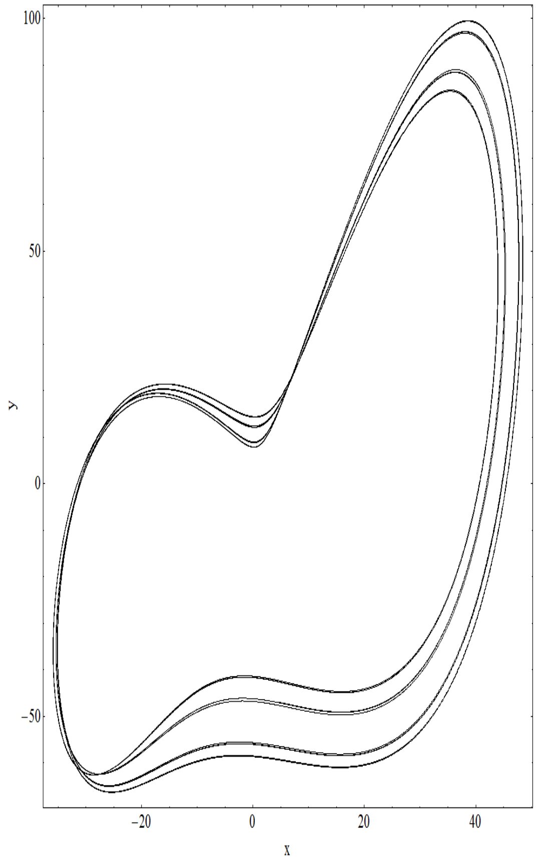


(f)

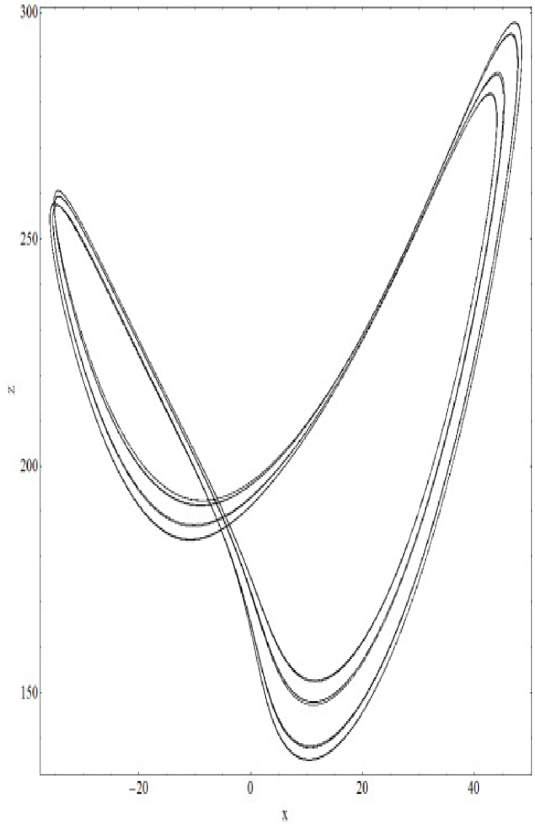
Figure 6.7: (a) Phase portraits of Lorenz system in the phase space when $\sigma = 10, b = \frac{8}{3}, r = 217$ with initial point $x = -8, y = 1, z = 24$ (b). Projection of (a) on the xy-plane (c). Trace of (b) in Poincare section (d).time series plot of x-values (e). time series plot of y-values (f). time series plot of z-values which shows period four behaviour.

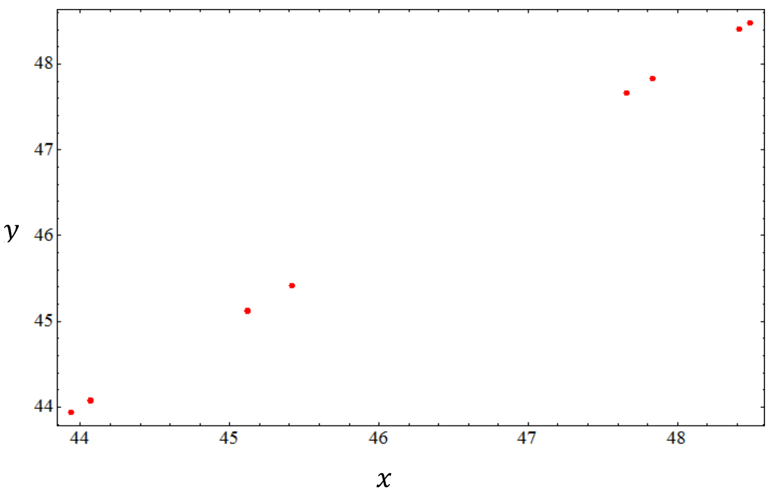


(a)

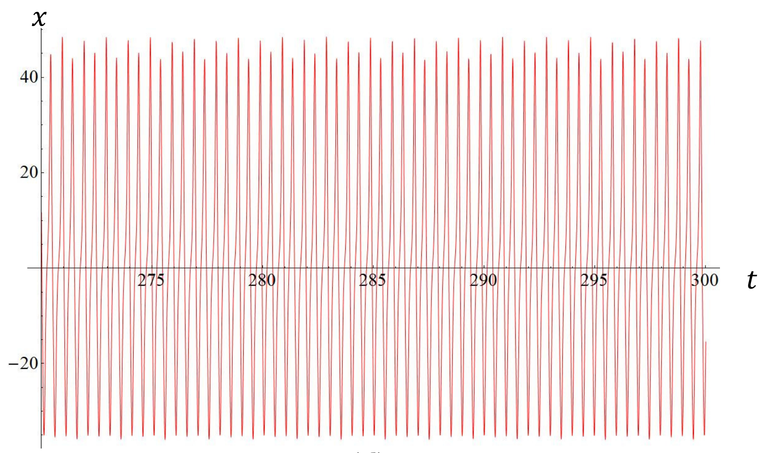

100
50
0
-50
y
-20
0
20
40
x

(b

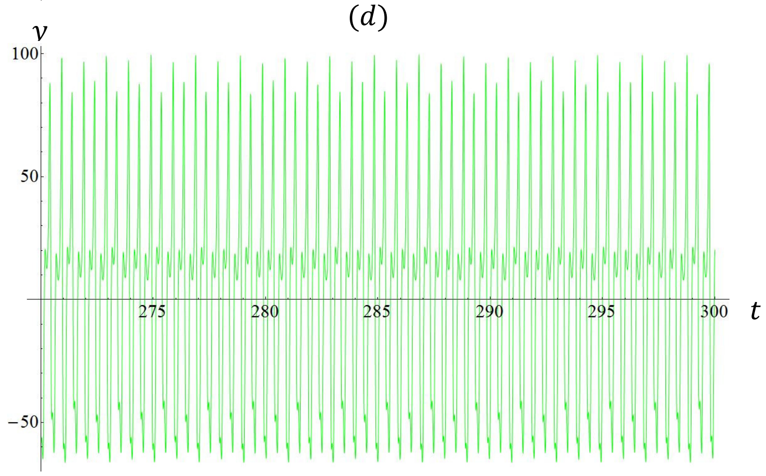

300
250
200
150
z
-20
0
20
40
x


48
47
46
45
44
y
44
45
46
47
48
x

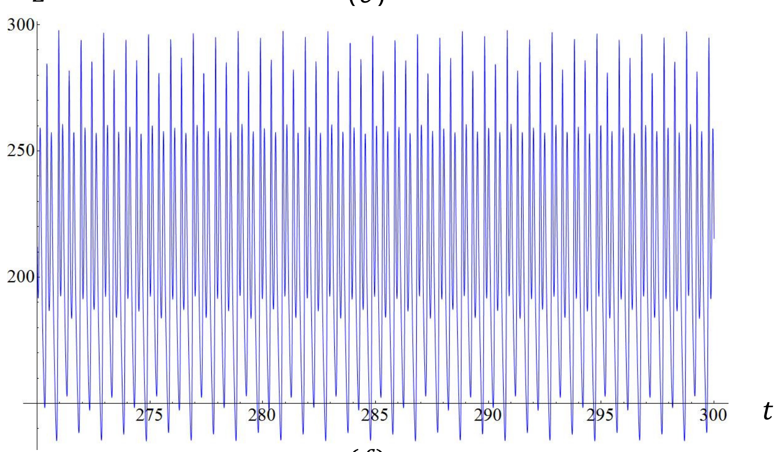
(c)



(d)

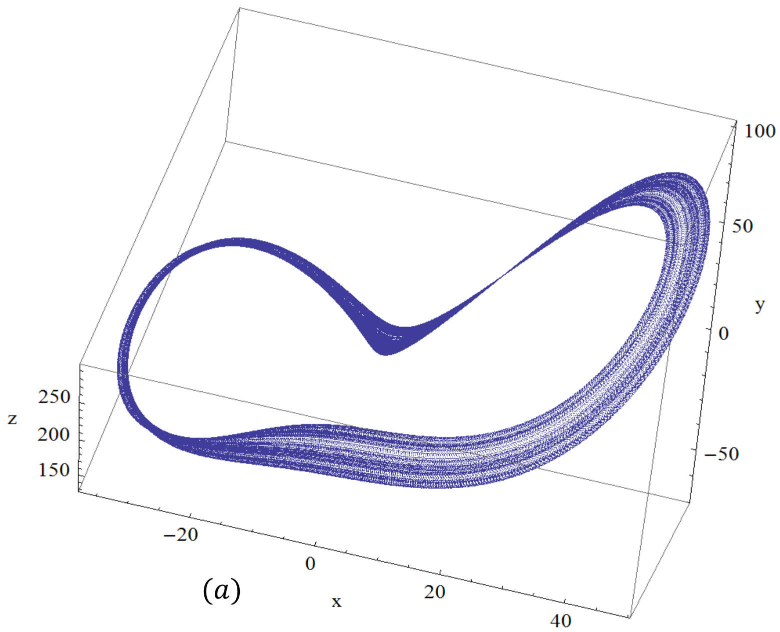


(e)

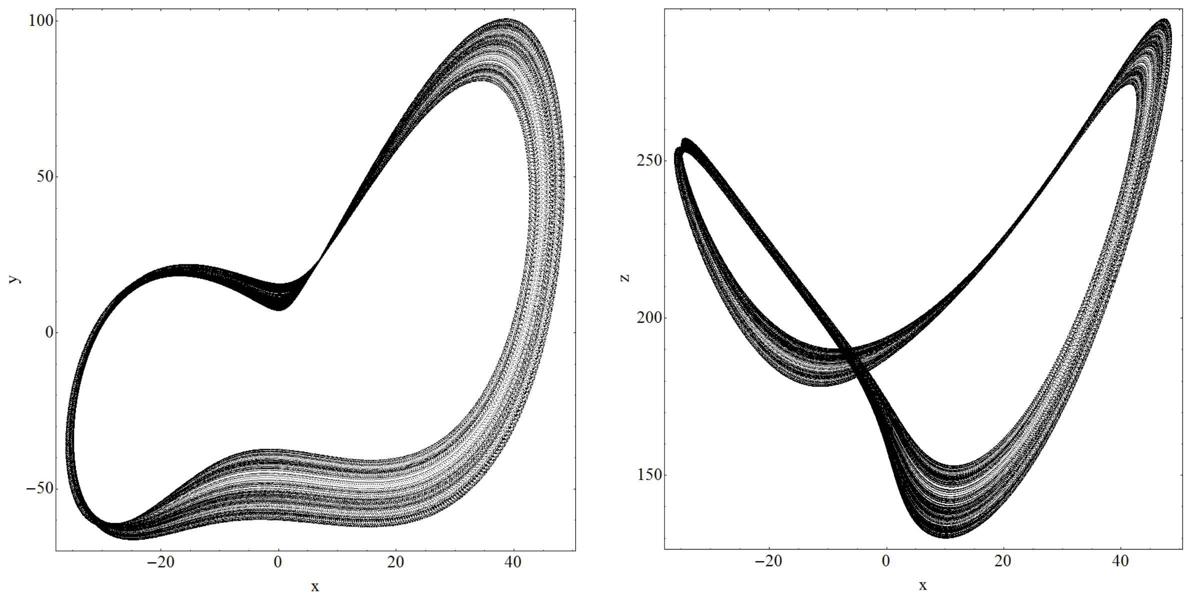


(f)

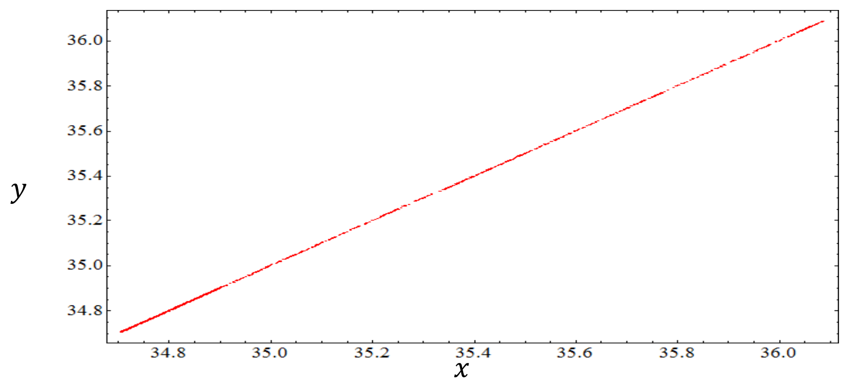
Figure 6.8: ***(a) Phase portraits of Lorenz system in the phase space when*** $\boldsymbol{\sigma = 10, b = \frac{8}{3}, r = 215.8}$ ***with initial point*** $\boldsymbol{x = -8, y = 1, z = 24}$ ***(b).Projection of (a) on the xy-plane and xz-plane (c).Trace of (b) in Poincare section (d).time series plot of x-values (e). time series plot of y-values (f). time series plot of z-values which shows period eight behaviour.***


100
50
y
0
−50
250
z
200
150
−20
0
x
20
40

(a)


100
50
y
0
−50
−20
0
20
40
x
250
z
200
150
−20
0
20
40
x

(b)


36.0
35.8
35.6
35.4
y
35.2
35.0
34.8
34.8
35.0
35.2
35.4
35.6
35.8
36.0
x

(c)

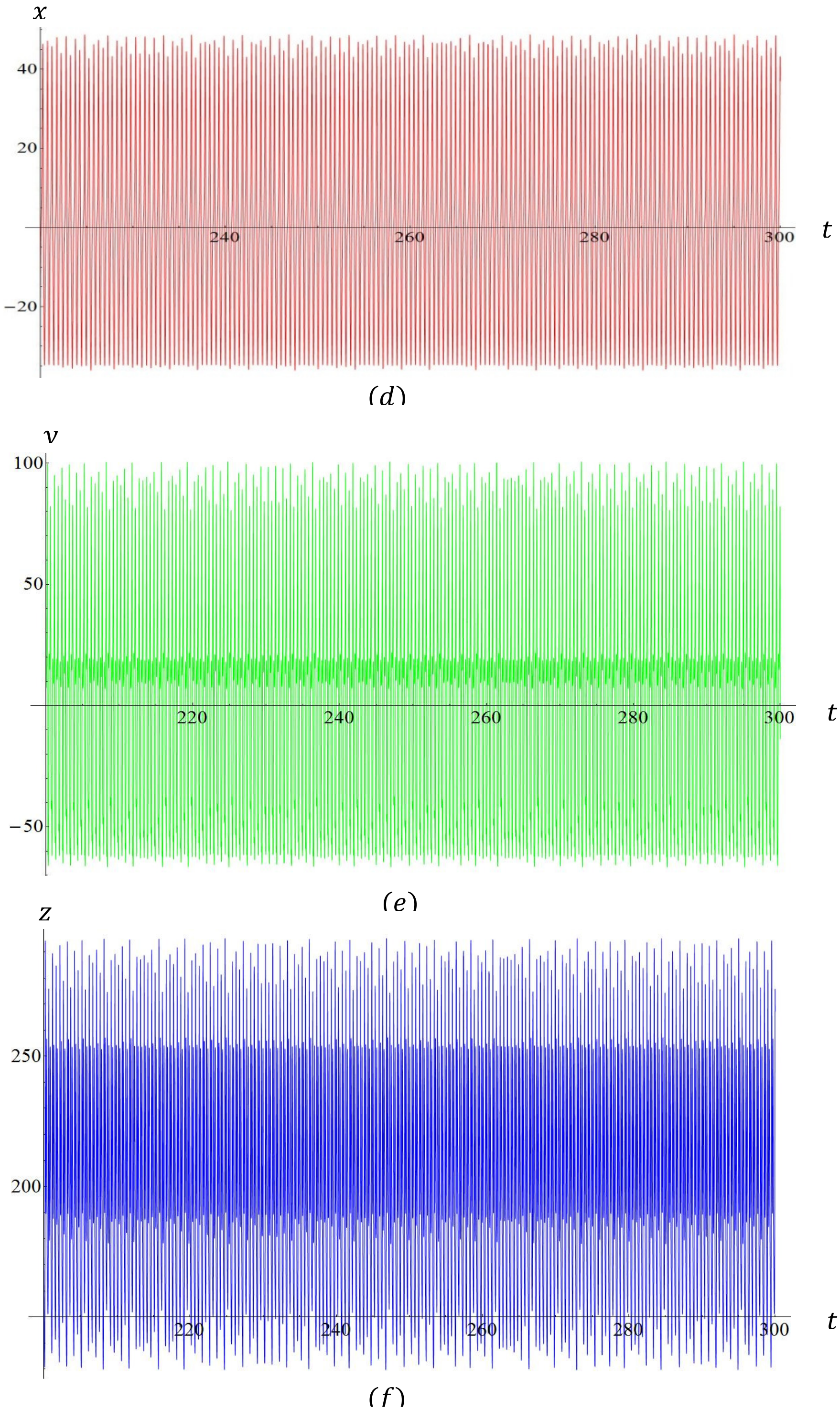


Figure 6.9: *$(\boldsymbol{a})$ Phase portraits of Lorenz system in the phase space when $\boldsymbol{\sigma} = \mathbf{10}, \boldsymbol{b} = \mathbf{8/3}, \boldsymbol{r} = \mathbf{212}$ with initial point $\boldsymbol{x} = -\mathbf{8}, \boldsymbol{y} = \mathbf{1}, \boldsymbol{z} = \mathbf{24}$ (b).Projection of (a) on the xy-plane and xz-plane (c).Trace of (b) in Poincare section (d).time series plot of x-values (e). time series plot of y-values (f). time series plot of z-values which shows the chaotic behaviour.*