Linearly Implicit Euler Method (Semi-Implicit Euler Method)

The **Linearly Implicit Euler method**, also known as the **semi-implicit Euler method**, is a numerical technique used to solve ordinary differential equations (ODEs) where the explicit Euler method might face stability issues. It is especially useful for stiff ODEs, which are equations where explicit methods require impractically small time steps to maintain numerical stability.

Overview of the Method

In the standard explicit Euler method, the next time step is calculated using the current value of the function's derivative, making it prone to instability in stiff problems. In contrast, the semi-implicit Euler method approximates the derivative in such a way that part of the calculation is implicit, and part is explicit, which improves stability.

Formula

For an ODE of the form:

$$\frac{dy}{dt} = f(y)$$

the explicit Euler method calculates the next step as:

$$y_{n+1} = y_n + \Delta t \cdot f(y_n)$$

In the linearly implicit Euler method, the function $f(y)$ is split into two parts: a linear part $A \cdot y$ and a non-linear part $N(y)$, such that:

$$f(y) = A \cdot y + N(y)$$

Then, the update for the next time step is computed as:

$$y_{n+1} = y_n + \Delta t \cdot (A \cdot y_{n+1} + N(y_n))$$

The implicit part $A \cdot y_{n+1}$ (the linear term) is solved for y_{n+1}, while the non-linear part $N(y_n)$ is computed explicitly using the value at the current time step y_n.

Applications

The linearly implicit Euler method is useful for stiff ODEs commonly found in physical simulations (e.g., fluid dynamics, chemical reaction kinetics). It helps improve stability without the heavy computational cost of a fully implicit method, making it a good compromise between efficiency and stability.