Problem

Solve $u_t = u_{xx}$ with boundary conditions $u(-1,t) = 0, u(1,t) = 0$ and initial conditions $u(x,0) = f(x)$.

Solution

Let $u = X(x) T(t)$. By separation of variables we obtain the following two ODEs. The spatial ODE is

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \qquad (1) \\ X(-1) &= 0 \\ X(1) &= 0 \end{aligned}$$

And the time ODE

$$T'(t) + \lambda T = 0 \qquad (2)$$

We can quickly see that $\lambda > 0$ is only possible case. Hence solution to (1) is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right) \qquad (2A)$$

Applying boundary conditions $X(-1) = 0$ gives

$$\begin{aligned} 0 &= A\cos\left(-\sqrt{\lambda}\right) + B\sin\left(-\sqrt{\lambda}\right) \\ &= A\cos\sqrt{\lambda} - B\sin\sqrt{\lambda} \qquad (3) \end{aligned}$$

Applying boundary conditions $X(1) = 0$ gives

$$0 = A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} \qquad (4)$$

(4),(3) give the system

$$\begin{pmatrix} \cos\sqrt{\lambda} & -\sin\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (5)$$

For non-trivial solution we want

$$\begin{aligned} \begin{vmatrix} \cos\sqrt{\lambda} & -\sin\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{vmatrix} &= 0 \\ \cos\sqrt{\lambda}\sin\sqrt{\lambda} + \cos\sqrt{\lambda}\sin\sqrt{\lambda} &= 0 \\ 2\cos\sqrt{\lambda}\sin\sqrt{\lambda} &= 0 \\ \cos\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) &= 0 \end{aligned}$$

But $\cos\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = \frac{1}{2}\sin\left(2\sqrt{\lambda}\right)$, hence $\frac{1}{2}\sin\left(2\sqrt{\lambda}\right) = 0$ or $2\sqrt{\lambda} = n\pi$ for $n = 1, 2, \cdots$. Therefore

$$\sqrt{\lambda_n} = \frac{n\pi}{2} \qquad n = 1, 2, \cdots$$

Each eigenvalue has associated eigenvector. For $n = 1$, (5) becomes

$$\begin{aligned} \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

First equation gives $-B_1 = 0$. Hence $B_1 = 0$ and A_1 can be anything. Hence first eigenfunction from (2A) is

$$X_1(x) = A_1\cos\left(\frac{\pi}{2}x\right)$$

For $n = 2$ (5) becomes

$$\begin{pmatrix} \cos\pi & -\sin\pi \\ \cos(\pi) & \sin(\pi) \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First equation gives $-A_2 = 0$. Hence $A_2 = 0$ and B_2 can be anything. Hence second eigenfunction from (2A) is

$$X_2(x) = B_2 \sin(\pi x)$$

We continue this way and find that for $n = 1, 3, 5, \cdots$ the eigenfunctions are

$$X_n(x) = A_n \cos\left(n\frac{\pi}{2}x\right)$$

And for $n = 2, 4, 6, \cdots$ the eigenfunctions are

$$X_n(x) = B_n \sin\left(n\frac{\pi}{2}x\right)$$

Therefore the spatial solution is

$$\begin{aligned} X(x) &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(n\frac{\pi}{2}x\right) + \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(n\frac{\pi}{2}x\right) \\ &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) + \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) \end{aligned}$$

The solution to the time domain ODE is $T(t) = e^{-\lambda_n t}$, therefore the complete solution is

$$\begin{aligned} u_n(x,t) &= X_n T_n \\ u(x,t) &= \sum_{n=0}^{\infty} X_n T_n \\ &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-\lambda_n t} + \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-\lambda_n t} \end{aligned}$$

To find A_n, we apply orthogonality. At $t = 0$. For $n = 1, 3, \cdots$ case

$$\begin{aligned} f(x) &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) \\ \int_{-1}^{1} f(x) \cos\left(\sqrt{\lambda_m}x\right) dx &= \int_{-1}^{1} \left(\sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right)\right) \cos\left(\sqrt{\lambda_m}x\right) dx \\ &= A_m \int_{-1}^{1} \cos^2\left(\sqrt{\lambda_m}x\right) dx \\ &= A_m \end{aligned}$$

To find B_n, we apply orthogonality. At $t = 0$. For $n = 2, 4, \cdots$ case

$$
\begin{aligned}
f(x) &= \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right)\\
\int_{-1}^{1} f(x)\sin\left(\sqrt{\lambda_m} x\right) dx &= \int_{-1}^{1}\left(\sum_{n=1,3,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right)\right)\sin\left(\sqrt{\lambda_m} x\right) dx\\
&= B_m \int_{-1}^{1} \sin^2\left(\sqrt{\lambda_m} x\right) dx\\
&= B_m
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n t} + \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n t} \qquad (6)\\
\sqrt{\lambda_n} &= \frac{n\pi}{2} \qquad n = 1,2,\cdots\\
A_n &= \int_{-1}^{1} f(x)\cos\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1,3,5,\cdots\\
B_n &= \int_{-1}^{1} f(x)\sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 2,4,6,\cdots
\end{aligned}
$$

To verify
Example 1 here is solution for $f(x) = 1 - x^2$ which satisfies boundary conditions. Using this we find

$$
\begin{aligned}
A_n &= \int_{-1}^{1} \left(1 - x^2\right)\cos\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1,3,5,\cdots\\
&= \frac{-16}{(n\pi)^3}\left(n\pi \cos\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

And

$$
\begin{aligned}
B_n &= \int_{-1}^{1} \left(1 - x^2\right)\sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 2,4,6,\cdots\\
&= 0
\end{aligned}
$$

Hence analytical solution (6) is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n}\pi x\right) e^{-\lambda_n t}\\
A_n &= \frac{-16}{(n\pi)^3}\left(n\pi \cos\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right)\right)\\
\sqrt{\lambda_n} &= \frac{n\pi}{2} \qquad n = 1,3,5,\cdots
\end{aligned}
$$

Example 2 $f(x) = \left(1 - x^2\right) x$ which satisfies boundary conditions. Using this we find

$$
\begin{aligned}
A_n &= \int_{-1}^{1} \left(1 - x^2\right) x \cos\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1,3,5,\cdots\\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
B_n &= \int_{-1}^{1} \left(1-x^2\right) x \sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 2,4,6,\cdots \\
&= \frac{-16}{(n\pi)^4}\left(6n\pi\cos\left(\frac{n\pi}{2}\right) + \left(-12+n^2\pi^2\right)\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

Hence analytical solution (6) is

$$
\begin{aligned}
u(x,t) &= \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n}\pi x\right) e^{-\lambda_n t} \\
B_n &= \frac{-16}{(n\pi)^4}\left(6n\pi\cos\left(\frac{n\pi}{2}\right) + \left(-12+n^2\pi^2\right)\sin\left(\frac{n\pi}{2}\right)\right) \\
\sqrt{\lambda_n} &= \frac{n\pi}{2} \qquad n = 2,4,\cdots
\end{aligned}
$$

Example 3

$f(x) = (1-x^2)(x+1)$ which satisfies boundary conditions. Using this we find

$$
\begin{aligned}
A_n &= \int_{-1}^{1} (1-x^2)(x+1)\cos\left(\sqrt{\lambda_n} x\right) dx \qquad n = 1,3,5,\cdots \\
&= \frac{-16}{(n\pi)^3}\left(n\pi\cos\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

And

$$
\begin{aligned}
B_n &= \int_{-1}^{1} (1-x^2)(x+1)\sin\left(\sqrt{\lambda_n} x\right) dx \qquad n = 2,4,6,\cdots \\
&= \frac{-16}{(n\pi)^4}\left(6n\pi\cos\left(\frac{n\pi}{2}\right) + \left(-12+n^2\pi^2\right)\sin\left(\frac{n\pi}{2}\right)\right)
\end{aligned}
$$

Hence analytical solution (6) is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1,3,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n t} + \sum_{n=2,4,\cdots}^{\infty} B_n \sin\left(\sqrt{\lambda_n} x\right) e^{-\lambda_n t} \\
A_n &= \frac{-16}{(n\pi)^3}\left(n\pi\cos\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right)\right) \\
B_n &= \frac{-16}{(n\pi)^4}\left(6n\pi\cos\left(\frac{n\pi}{2}\right) + \left(-12+n^2\pi^2\right)\sin\left(\frac{n\pi}{2}\right)\right) \\
\sqrt{\lambda_n} &= \frac{n\pi}{2} \qquad n = 1,2,3,\cdots
\end{aligned}
$$