Solve the heat equation $u_t = ku_{xx}$ with periodic boundary conditions $u(t,-\pi) = u(t,\pi), u_x(t,-\pi) = u_x(t,\pi)$

Solution

Using separation of variables, Let $u(x,t) = T(t) X(x)$. Substituting this into $u_t = ku_{xx}$ gives $T'X = TX''$. Dividing by $XT \neq 0$ gives

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Where λ is the seperation constant. This gives the following ODE's to solve

$$X''(x) + \lambda X(x) = 0$$
$$T'(t) + \lambda k T(t) = 0$$

Where λ is the separation constant. Eigenfunctions are solutions to the spatial ODE.

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

To determine the actual eigenfunctions and eigenvalues, boundary conditions are used. Starting with the spatial ODE above, and transferring the boundary condition to X, it becomes

$$X''(x) + \lambda X(x) = 0$$
$$X(-\pi) = X(\pi)$$
$$X'(-\pi) = X'(\pi)$$

This is an eigenvalue boundary value problem. The solution is

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \tag{1}$$

case $\lambda < 0$

Since $\lambda < 0$, then $-\lambda$ is positive. Let $\mu = -\lambda$, where μ is now positive. The solution (1) becomes

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

The above can be written as

$$X(x) = c_1 \cosh(\sqrt{\mu}x) + c_2 \sinh(\sqrt{\mu}x) \tag{2}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using (2) gives

$$c_1 \cosh(\sqrt{\mu}\pi) + c_2 \sinh(-\sqrt{\mu}\pi) = c_1 \cosh(\sqrt{\mu}\pi) + c_2 \sinh(\sqrt{\mu}\pi)$$
$$c_2 \sinh(-\sqrt{\mu}\pi) = c_2 \sinh(\sqrt{\mu}\pi)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_2 = 0$. The solution (2) now reduces to

$$X(x) = c_1 \cosh(\sqrt{\mu}x) \tag{3}$$

Taking derivative gives

$$X'(x) = c_1 \sqrt{\mu} \sinh(\sqrt{\mu}x) \tag{4}$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using (4) gives

$$c_1 \sqrt{\mu} \sinh(-\sqrt{\mu}\pi) = c_1 \sqrt{\mu} \sinh(\sqrt{\mu}x)$$

But sinh is only zero when its argument is zero which is not the case here. Therefore the above implies that $c_1 = 0$. This means a trivial solution. Therefore $\lambda < 0$ is not an eigenvalue.

case $\lambda = 0$

In this case the solution is $X(x) = c_1 + c_2 x$. Applying first BC $X(-\pi) = X(\pi)$ gives

$$c_1 - c_2\pi = c_1 + c_2\pi$$
$$-c_2\pi = c_2\pi$$

This gives $c_2 = 0$. The solution now becomes $X(x) = c_1$ and $X'(x) = 0$. Applying the second boundary conditions $X'(-\pi) = X'(\pi)$ is not satisfies $(0 = 0)$. Therefore $\underline{\lambda = 0}$ is an eigenvalue with eigenfunction $X_0(0) = 1$ (selected $c_1 = 1$ since an arbitrary constant).

case $\lambda > 0$

The solution in this case is

$$\begin{aligned} X(x) &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \end{aligned}$$

Which can be rewritten as (the constants c_1, c_2 below will be different than the above c_1, c_2, but kept the same name for simplicity).

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \tag{5}$$

Applying first B.C. $X(-\pi) = X(\pi)$ using the above gives

$$\begin{aligned} c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_1 \cos\left(\sqrt{\lambda}\pi\right) + c_2 \sin\left(\sqrt{\lambda}\pi\right) \\ c_2 \sin\left(-\sqrt{\lambda}\pi\right) &= c_2 \sin\left(\sqrt{\lambda}\pi\right) \end{aligned}$$

There are two choices here. If $\sin\left(-\sqrt{\lambda}\pi\right) \neq \sin\left(\sqrt{\lambda}\pi\right)$, then this implies that $c_2 = 0$. If $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$ then $c_2 \neq 0$. Assuming for now that $\sin\left(-\sqrt{\lambda}\pi\right) = \sin\left(\sqrt{\lambda}\pi\right)$. Then happens when $\sqrt{\lambda}\pi = n\pi, n = 1, 2, 3, \cdots$, or

$$\lambda_n = n^2 \qquad n = 1, 2, 3, \cdots$$

Using this choice, we will now look to see what happens using the second BC. The solution (5) now becomes

$$X(x) = c_1 \cos(nx) + c_2 \sin(nx) \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X'(x) = -c_1 n \sin(nx) + c_2 n \cos(nx)$$

Applying the second BC $X'(-\pi) = X'(\pi)$ using the above gives

$$\begin{aligned} c_1 n \sin(n\pi) + c_2 n \cos(n\pi) &= -c_1 n \sin(n\pi) + c_2 n \cos(n\pi) \\ c_1 n \sin(n\pi) &= -c_1 n \sin(n\pi) \\ 0 &= 0 \end{aligned}$$

Since n is integer. Therefore this means that using $\lambda_n = n^2$ will satisfy both boundary conditions with $c_2 \neq 0, c_1 \neq 0$. This means the solution (5) becomes

$$X_n(x) = A_n \cos(nx) + B_n \sin(nx) \qquad n = 1, 2, 3, \cdots$$

The above says that there are two eigenfunctions in this case. They are

$$X_n(x) = \begin{cases} \cos(nx) \\ \sin(nx) \end{cases}$$

Since there is also zero eigenvalue, then the complete set of eigenfunctions become

$$X_n(x) = \begin{cases} 1 \\ \cos(nx) \\ \sin(nx) \end{cases}$$

Now that the eigenvalues are found, the solution to the time ODE can be found. Recalling that the time ODE from above was found to be

$$T'(t) + \lambda k T(t) = 0$$

For the zero eigenvalue case, the above reduces to $T'(t) = 0$ which has the solution $T_0(t) = C_0$. For non zero eigenvalues $\lambda_n = n^2$, the ODE becomes $T'(t) + n^2 T(t) = 0$, whose solution is $T_0(t) = C_n e^{-kn^2 t}$.

Putting all the above together, gives the fundamental solution as

$$u_n(x,t) = \begin{cases} C_0 & \\ C_n \cos(nx)\, e^{-kn^2 t} & n = 1, 2, 3, \cdots \\ B_n \sin(nx)\, e^{-kn^2 t} & n = 1, 2, 3, \cdots \end{cases}$$

Therefore the complete solution is the sum of the above solutions

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} e^{-kn^2 t} \left(C_n \cos(nx) + B_n \sin(nx) \right)$$

The constants C_0, C_n, B_n can be found from initial conditions.