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Force vibration of 1-D :

$$\mu(x)\, w_{,tt} + K[w] = q(x,t)$$

$q(x,t)$ → forcing term

$w(0,t) = 0$ $\quad$ $w(l,t) = 0$

$w(x,0) = w_0(x)$ $\quad$ $w_t(x,0) = v_0(x)$

$q(x,t)$ → Harmonic forcing
$q(x,t)$ → General forcing

For Harmonic forcing →

$$q(x,t) = Q(x)\cos(\Omega t)$$

where, Ω = forcing circular frequency

* If we know Harmonic forcing we can also deduce for periodic forcing.

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$$\mu(x) w_{tt} + K(w) = R[Q(x) e^{i\Omega t}]$$

$$W(x,t) = W_h(x,t) + W_p(x,t)$$

where,

$$W_h(x,t) = \sum_{k=1}^{\infty} [C_k \cos\omega_k t + S_k \sin\omega_k t] \times W_k(x)$$

eigen function

$$W_p(x,t) = R[X(x) e^{i\Omega t}]$$

$$-\Omega^2 \mu(x) X(x) + K[X(x)] = Q(x).$$

$$X(0) = 0 \qquad X(l) = 0$$

Eigen function Expansion Method:

$$X(x) = \sum_{k=1}^{\infty} \alpha_k W_k(x)$$ where α_k = constants (By solving eqn)

$$-\Omega^2 \mu(x) X(x) + K[X(x)] = Q(x)$$

or, $$-\Omega^2 \mu(x) X(x) + K X(x) = Q(x)$$

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$$-\Omega^2 \mu(x) \sum \alpha_k W_k(x) + k\left[\sum \alpha_k W_k(x)\right] = Q(x)$$

or, $$-\Omega^2 \mu(x) \sum \alpha_k W_k(x) + \sum \alpha_k W_k(x) \cdot k = Q(x) \quad \text{--- (1)}$$

eigen-value problem $\rightarrow$

$$-\omega_k^2 \mu(x) W_k(x) + k W_k(x) = 0$$

or, $$k W_k(x) = \omega_k^2 \mu(x) W_k(x) \quad \text{--- (2)}$$

from (1), (2) $\rightarrow$

$$-\Omega^2 \mu(x) \sum \alpha_k W_k(x) + \sum \alpha_k \omega_k^2 \mu(x) W_k(x) = Q(x)$$

or, $$\sum \alpha_k W_k(x) \left(\omega_k^2 \mu(x) - \Omega^2 \mu(x)\right) = Q(x)$$

or, $$\sum_{n=1}^{\infty} \mu(x)(\omega_k^2-\Omega^2)(\alpha_k W_k(x)) = Q(x)$$

From orthogonality ~~function~~ condition.

$$\underbrace{\int_0^l \mu(x) W_j(x) W_k(x)\, dx = 0}_{\langle W_j, W_k\rangle = 0} \quad \text{for } j = k$$

for $j = k$

Take inner product with $W_j(x)$.

$$(\omega_j^2-\Omega^2)\,\alpha_j \int_0^l \mu(x) W_j^2(x)\, dx$$

$$= \int_0^l Q(x) W_j(x)\, dx.$$

or, $$\alpha_j = \frac{\int_0^l Q(x) W_j(x)\, dx}{(\omega_j^2-\Omega^2)\int_0^l \mu(x) W_j^2(x)\, dx}$$

$\boxed{\text{and } \Omega \neq \omega_j}$

 where $j = 1, 2, 3$

so, for non-resonant case,

$$X(x) = \sum_{k=1}^{\infty} \alpha_k W_k(x)$$

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$$W_p(x,t) = X(x)\cos(\Omega t)$$

$$\boxed{W(x,t) = W_r(x,t) + W_p(x,t)}$$

(complete solution)

Initial Conditions $\left.\begin{array}{l} w(x,0) = W_0(x) \\ w_t(x,0) = V_0(x) \end{array}\right\}$ To solve the C_k and S_k.

Free Vibration →

$$\mu(x)\, w_{tt} + k(W) = 0 \qquad w(x,t) = u(x)\, S(t)$$

or $\mu(x) \dfrac{\partial^2 W(x,t)}{\partial t^2} + k\, W(x,t) = 0$

or $\dfrac{\partial^2 W(x,t)}{\partial t^2} + \dfrac{k}{\mu(x)} W(x,t) = 0$

or, $u(x)\left(\dfrac{d^2 S}{dt^2} + \dfrac{k}{\mu(x)} S(t)\right) = 0$

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$$\frac{d^2 s}{dt^2} + \frac{k}{\mu(x)} \cdot s(t) = 0$$

or, $$\frac{d^2 s}{dt^2} + \omega^2 s(t) = 0$$

or $$\left(D^2 + \omega^2\right) s = 0$$

$$s(t) = \left[c \cos(\omega t) + s \sin(\omega t)\right]$$

$$w(x,t) = u(x) \left[c \cos(\omega t) + s \sin(\omega t)\right]$$

$\boxed{\text{Detailed Solution}}$ →

$$\mu(x)\, u_{tt} + \mathcal{L}[u] = 0$$

$$u(0,t) = 0, \quad u_x(l,t) = 0$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

$$u(x,t) = \sum_{k=1}^{\infty} p_k(t)\, U_k(x)$$

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$$\mu(x) \sum_{k=1}^{\infty} \ddot{p}_k U_k(x) + K\left[\sum_{k=1}^{\infty} p_k U_k(x)\right] = 0$$

$$\text{or,} \quad \sum_{k=1}^{\infty} \ddot{p}_k \mu(x) U_k(x) + \sum_{k=1}^{\infty} p_k K U_k(x) = 0 \quad \text{--- } (1)$$

Eigen value problem →

$$-\omega_k^2 \mu(x) U_k(x) + K U_k(x) = 0$$

$$U(0) = 0 \quad \text{and} \quad U'(l) = 0 \quad \text{--- } (2)$$

from (2) and (1)

$$\sum_{k=1}^{\infty} \left[\ddot{p}_k + \omega_k^2 p_k\right] \mu(x) U_k(x) = 0$$

inner product with $U_j(x)$

$$\ddot{p}_j + \omega_j^2 p_j = 0 \qquad \text{where, } j = 1, 2, 3 \ldots \infty$$

$$p_j(t) = C_j \cos(\omega_j t) + S_j \sin(\omega_j t)$$

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$$u(x,t) = \sum_{j=1}^{\infty} \left(c_j \cos(\omega_j t) + s_j \sin(\omega_j t) \right) v_j(x)$$ --- (3)

Initial conditions →

$$u(x,0) = u_0(x)$$
$$u_t(x,0) = v_0(x)$$

$$\sum_{j=1}^{\infty} c_j (1) \cdot v_j(x) = u_0(x)$$

$$\sum_{j=1}^{\infty} s_j \omega_j v_j(x) = v_0(x)$$

By taking inner product →

$$c_k \langle v_k, v_k \rangle = \langle u_0 \cdot v_k \rangle$$

$$\Rightarrow c_k = \frac{\int_0^L u_0(x) v_k(x)\, dx}{\int_0^L v_k^2(x)\, dx}$$

where $k = 1, 2, 3$ —

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$S_k N_k \langle U_k, U_k \rangle = \langle V_0(x), U_k(x) \rangle$

$$S_k = \frac{\langle V_0(x)\, U_k(x) \rangle}{w_k \langle U_k, U_k \rangle}$$

$$= \frac{1}{w_k} \cdot \frac{\int_0^l V_0(x)\, U_k(x)\, dx}{\int_0^l U_k^2(x)\, dx}$$