From the equation of motion for the forced lateral vibration of a nonuniform beam —

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial t^2}(x,t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t). \quad \text{eq(9.B)}$$

For a uniform beam, above equation reduces to

$$\boxed{EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t).} \quad \text{— (9.B.1)}$$

\# eq^n of forced vibration of Euler-Bernoulli's Beam equation

eq^n (9.B.1) can be reduced to free vibration equation when $f(x,t) = 0$ — (9.B.2)

$$EI.\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = 0$$

where, $C = \sqrt{\frac{EI}{\rho A}}$

<u>Initial Conditions:</u>

For finding a unique solution for $w(x,t)$, two initial conditions and four boundary conditions are needed as per the terms present in the equation.

$$w(x, t=0) = w_0(x) \quad \text{— (9.B.3.1)}$$

$$\frac{\partial w}{\partial t}(x, t=0) = \dot{w}_0(x) \quad \text{— (9.B.3.2)}$$

The free-vibration solution can be found using the method of separation of variables as –

$$w(x,t) = W(x)\,T(t) \quad \text{--- (9.B.4)}$$

Substituting Eq. (9.B.4) in Eq. (9.B.2) and rearranging leads to –

$$\frac{c^2}{W(x)} \cdot \frac{d^4 W(x)}{dx^4} = -\frac{1}{T(t)} \cdot \frac{d^2 T(t)}{dt^2} = a = \omega^2 = \text{+ve constant}$$

$$\text{--- (9.B.5)}$$

Eqn (9.B.5) can be written as two equations:

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0 \quad \text{where, } \beta^4 = \frac{\omega^2}{c^2}$$

$$\text{--- (9.B.6)} \quad \text{or, } \beta^4 = \frac{\rho A \omega^2}{EI}$$

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \quad \text{--- (9.B.7)}$$

Eq. (9.B.7) $\rightarrow$

$$(D^2 + \omega^2)\,T = 0$$

$$\text{or, } D_{1,2} = \pm \omega i$$

eqn (9.B.7). $\rightarrow$ solution can be written as –

$$T(t) = e^{0.x}\left(A\cos\omega t + B\sin(\omega t)\right) \quad \text{--- (9.B.8)}$$

$$\text{or, } T(t) = A\cos(\omega t) + B\sin(\omega t)$$

(A, B are constants)

From the intial condition we can calculate A, B

For the solution of Eq (9.B.9) –
assume – $W(x) = C e^{sx}$.

where, C and s are constants and derive the auxiliary eq^n as –

$$s^4 - \beta^4 = 0 \qquad \text{— (9.B.10)}$$

$$\text{or,} \quad (s^2+\beta^2)(s^2-\beta^2) = 0$$

$$s_{1,2} = \pm \beta i \qquad s_{3,4} = +\beta, -\beta$$

Hence, the solution becomes –

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}.$$

$$\text{or,} \quad W(x) = B_1 \cos(\beta x) + B_2 \sin(\beta x) + B_3 \cosh(\beta x) + B_4 \sinh(\beta x) \qquad \text{— (9.B.11)}$$

where, B_1, B_2, B_3, B_4 are different constants. are real constants of integration i.e. B_i, $i = 1,2,3,4$ are determined by the boundary condition which are of the problem.

For typical support conditions, we are going to calculate the corresponding eignfunctions and eigen frequencies.

(a) Uniform Simply Supported Beam:

For a pinned-pinned Euler bernouli's beam, the conditions for the corresponding eigen value problems are –

$$W(0) = 0 \qquad W(l) = 0$$
$$W''(0) = 0 \qquad W''(l) = 0$$

$B_1 + B_3 = 0$ — (9.B.12)

$W(x) = B_1 \cos(Bx) + B_2 \sin(Bx) + B_3(\cosh Bx) + B_4 \sinh(Bx)$

or, $W'(x) = -B_1 B \sin Bx + B_2 B \cos(Bx) + B_3 B \sinh Bx + B_4 B \cosh Bx$

or, $W''(x) = -B_1 B^2 \cos Bx + B_2(-B^2)\sin(Bx) + B_3 B^2 \cosh Bx + B_4 B^2 \sinh Bx$

$\therefore W''(0) = -B_1 B^2 + 0 + B_3 B^2 + 0$

or, $-B_1 + B_3 = 0$

or, $B_1 - B_3 = 0$ — (9.B.13)

By solving those eqn:

$B_1 = B_3 = 0$

$\therefore W(x) = B_2 \sin(Bx) + B_4 \sinh(Bx)$

or, $W(l) = B_2 \sin(Bl) + B_4 \sinh(Bl)$

or, $B_2 \sin(Bl) + B_4 \sinh(Bl) = 0$ — (9.B.14)

$B_2 \sin(Bl) - B_4 \sinh(B l) = 0$ — (9.B.15)

For non-trivial solution of (B_2, B_4) -

$$\underbrace{\begin{bmatrix} \sin(Bl) & \sinh(Bl) \\ \sin(Bl) & -\sinh(Bl) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} B_2 \\ B_4 \end{bmatrix}}_{X} = \underbrace{0}_{B}$$

$\therefore |A| = 0$

or, $\sin(Bl)\sinh(Bl) = 0$

or, $\sin Bl = 0$ (Since $\sinh Bl \neq 0$ for any $Bl \neq 0$)

$\boxed{\sin(\beta_n l) = 0}$ → characteristic equation of the problem.

The solution of the characteristic equation for the problem can be obtained as —

$$\beta_n = \left(\frac{n\pi}{l}\right) \quad \text{where, } n = 1, 2, 3, \ldots \infty$$

we know that,

$$\beta^4 = \frac{\rho A \omega^2}{EI}$$

$$\text{or, } \frac{n^4\pi^4}{l^4} = \frac{\rho A \omega^2}{EI}$$

$$\text{or, } \omega^2 = \frac{n^4\pi^4(EI)}{\rho A(l^4)}$$

$$\text{or, } \omega = \frac{n^2\pi^2}{l^2} \cdot \sqrt{\frac{EI}{\rho A}} \quad \text{where, } n = 1, 2, \ldots \infty$$

where, ω = the circular natural frequencies of simply supported Euler-Bernoulli beam.

Hence, eigen function for simply-supported beam can be written as —

$$W(x) = \cancel{B_1}^{0} \cos(\beta_r x) + B_2 \sin(\beta_r x) + \cancel{B_3}^{0} \cosh(\beta_r x) + \cancel{B_4}^{0} \sinh(\beta_r x)$$

$$\text{or, } W(x) = B_2 \sin(\beta_r x)$$

$$\text{or, } \boxed{W_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{— (2.B.16)}$$

⇒ The first three normal modes normalized so that $A_n = 1$ ($n = 1, 2, 3$), are plotted in figure.

Therefore, substituti...

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The initial value problem can be calculated as –

$$w(x,t) = \sum_{j=1}^{\infty} \left(A_j \cos(\omega_j t) + B_j \sin(\omega_j t)\right) W_j(x) \quad \text{— (9.B.18)}$$

$$\omega_1 = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A^2}} \;;\; \omega_2 = \frac{4\pi^2}{l^2}\sqrt{\frac{EI}{\rho A^2}} \;;\; \omega_3 = \frac{9\pi^2}{l^2}\sqrt{\frac{EI}{\rho A^2}}$$

From initial position and velocity conditions as

$w(x,0) = w_0(x)$

$\dot{w}(x,0) = V_0(x)$

where, $W_j(x)$ = the eigenfunctions of the beam

A_j and B_j = unknown constants which are to be determined from intial conditions

from eqn (9.B.18) —

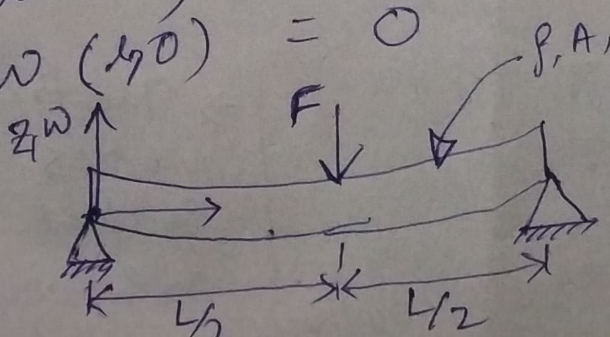
$$w(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \sin\left(\frac{n\pi x}{l}\right) \quad \text{— (9.B.19)}$$

where $w(x,0) = w_0(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$

$$\frac{dw(x,0)}{dt} = V_0(x) = 0$$

The initial shape $w_0(x)$ should be satisfied by –

$$\left.\begin{aligned} EI\, w_{,xxxx}(x,0) &= -F\delta\left(x-\frac{L}{2}\right) \\ w(0,0) &= 0 \\ w(L,0) &= 0 \end{aligned}\right\} \quad \text{— (9.B.20)}$$



Therefore, substituting the first condition from the above equations and taking the inner product of both sides with $\sin\left(\frac{m\pi x}{l}\right)$ yields on simplification -

$$C_m = \begin{cases} \frac{2Fl^3}{m^4\pi^4(EI)}(-1)^{\left(\frac{m-1}{2}\right)} ; & m = 1,3,5, \ldots \infty. \\ 0 ; & m = 2,4,6, \ldots \infty \end{cases}$$

Using the initial condition on the velocity $\dot{w}(x,0) \equiv 0$ one can easily obtain —

$$S_m = 0 ; \quad m = 1,2,3, \ldots \infty.$$

Thus, the solution of the initial value problem -

$$w(x,t) = \sum_{n=1,3,5}^{\infty} \left(\frac{2Fl^3}{n^4\pi^4EI}\right)(-1)^{(n-1)/2}\cos(\omega_n t)\sin\left(\frac{n\pi x}{l}\right) \quad (9.B.21)$$

⇒ Forced vibration analysis, The general forced vibration problem for an Euler-Bernoulli's beam can be represented as —

$$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = q(x,t) \quad (9.B.22)$$

along with corresponding boundary conditions

where, $q(x,t)$ = general forcing function

⇒ eigen function expansion method, (9.B.23)

$$w(x,t) = w_H(x,t) + w_p(x,t)$$

where, $w_H(x,t)$ = general solution to the homogeneous problem (i.e. $q(x,t) = 0$)

homogeneous problem

Consider a particular solution in the form of the eigenfunction expansion

$$w_p(x,t) = \sum_{j=1}^{\infty} p_j(t)\, W_j(x) \qquad (9.B.24)$$

$W_j(x)$ = eigenfunctions.

$p_j(t)$ = corresponding modal coordinates

(9.B.24) will then automatically fulfill the boundary conditions.

From eqn (9.B.23) and (9.B.24) — (9.B.25)

$$\sum_{j=1}^{\infty} \rho A \ddot{p}_j W_j + (EI W_j'''') p_j = q(x,t)$$

(9.B.26)

$$\text{or,} \quad \sum_{j=1}^{\infty} \rho A [\ddot{p}_j + \omega_j^2 p_j] W_j = q(x,t)$$

taking the inner product with $W_K(x)$ on both sides and using the orthonormality condition–

~~taking the inner produ.~~

$$\ddot{p}_K + \omega_K^2 p_K = f_K(t) \quad \text{where, } k = 1, 2, \ldots \infty. \qquad (9.B.27)$$

$$\text{where, } f_K(t) = \int_0^l q(x,t)\, W_K(x)\, dx.$$

Thus, (9.B.27) represents the modal dynamics of the forced Euler-Bernoulli beam. These equations can be solved using standard techniques such Laplace transform or Green's function method

The complete solution is then obtained from (3B8) as —

$$w(x,t) = \sum_{j=1}^{\infty} \left(A_j' \cos(\omega_j t) + B_j' \sin(\omega_j t)\right) W_j(x) + \sum_{j=1}^{\infty} p_j(t)\, W_j(x)$$

where, A_j' and B_j' are the constants of integration to be determined from the initial conditions.

$$f_k(t) = \underbrace{\int_0^l F_0 \sin(\Omega t)(1 - x/L)\, W_k(x)\, dx}$$ where, $k = 1, 2, 3, \ldots$

modal forces

or, $$f_k(t) = \left[\int_0^l W_k(x)\, F_0 \left(1 - \frac{x}{L}\right) dx\right] \cos(\Omega t) = F_k \cos(\Omega t)$$

where, $k = 1, 2, 3, \ldots$

and $$F_k = \int_0^l W_k(x)\, F_0 \left(1 - \frac{x}{L}\right) dx$$ = modal force amplitudes

$$p_k(t) = \frac{F_k}{\omega_k^2 - \Omega^2} \cos(\Omega t)$$ where, $k = 1, 2, \ldots$

∴ Steady-state harmonic response is —

$$w(x,t) = \left[\sum_{j=1}^{\infty} \frac{F_j}{\omega_j^2 - \Omega^2} \cdot W_j(x)\right] \cos(\Omega t).$$