Linear System of Equations. Let $\{y_1, y_2, y_3, y_4\}$ denote the **dependent** variables, $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ the **parameters**, and x the '**shocks**' (which we'll feed into the system to study the change of dependent variables for given parameters). Consider the following system of equations:

$$y_{1,i} = y_{2,i} - y_{3,i}, \tag{1}$$

where

$$\begin{aligned} y_{2,i} = \frac{1}{\delta_i}\sum_{j=1}^{2}\delta_j \Bigg[\alpha_{ij}(1-\gamma_j)\left(y_{2,j} - \varepsilon\sum_{k=1}^{2}\alpha_{kj}\left[-\frac{1}{\varepsilon}(x_i - x_k) + (\gamma_i y_{2,i} - \gamma_k y_{2,k}) + ((1-\gamma_i)y_{4,i} - (1-\gamma_k)y_{4,k})\right]\right) \\ + \beta_{ij}\gamma_j\left(y_{2,j} - \varepsilon\sum_{k=1}^{2}\beta_{kj}\left[-\frac{1}{\varepsilon}(x_i - x_k) + (\gamma_i y_{2,i} - \gamma_k y_{2,k}) + ((1-\gamma_i)y_{4,i} - (1-\gamma_k)y_{4,k})\right]\right)\Bigg], \end{aligned} \tag{2a}$$

and

$$y_{4,j} = \sum_{i=1}^{2}\alpha_{ij}\left(-\frac{1}{\varepsilon}x_i + \gamma_i y_{2,i}\right) + \sum_{i=1}^{2}\alpha_{ij}(1-\gamma_i)y_{4,i} \quad , \quad y_{3,j} = \sum_{i=1}^{2}\beta_{ij}\left(-\frac{1}{\varepsilon}x_i + \gamma_i y_{2,i}\right) + \sum_{i=1}^{2}\beta_{ij}(1-\gamma_i)y_{4,i}, \tag{3}$$

subject to the following parameter constraints:

$$\sum_{i=1}^{2}\alpha_{ij} = 1 \quad , \quad \sum_{i=1}^{2}\beta_{ij} = 1 \quad , \quad \delta_i = \sum_{j=1}^{2}(\alpha_{ij}(1-\gamma_j)\delta_j + \beta_{ij}\gamma_j\delta_j) \quad , \quad \gamma_i, \alpha_{ij}, \beta_{ij} \in (0,1), \quad \varepsilon, \delta_i, \delta_j > 0. \tag{4}$$

The equilibrium of this system is pinned down by two sets of equations – the y_2's from equation (2) and the y_4's from equation (3). Importantly, under the parameter constraints from equation (4), the system in equation (2) is homogenous of degree 1 and has infinitely many solutions. However, by choosing any of the y_2's as a scaling factor, it is possible to obtain an **up-to-scale unique** solution (see attached Maple file).

Matrix Form. It is possible to rewrite the above system of equations in the following (block) matrix form:

$$\mathbf{y_1} = X(I - M)^{+}Z + V, \tag{5}$$

where

- $\mathbf{y_1}$ is a 2×1 vector of the key dependent variables, $y_{1,1}$ and $y_{1,2}$,
- X is a 2×4 matrix of independent variables, I is a 4×4 identity matrix, M is a 4×4 matrix of independent variables, Z is a 4×1 vector of independent variables and V is a 2×1 vector of independent variables
- the $+$ sign stands for the Moore-Penrose pseudoinverse, as the matrix $I - M$ is singular (not full rank) and, hence, non-invertible under the parameter constraints (note that this is a direct implication of the fact that the system of y_2's from equation (2) has infinitely many solutions).

Importantly, as shown in the attached Maple file, the solution of equation (5) is identical to the one obtained from solving the original system of linear equations. Also, we know the left eigenvector (and the stationary distribution) of matrix M, which allows us to rewrite equation (5) in terms of a regular inverse:

$$\mathbf{y_1} = X(I - M)^{+}Z + V = \mathbf{y_1} = X(I - A)^{-1}Z + V, \tag{6}$$

where $A \equiv M + Q$ and Q is a suitably defined 4×4 matrix of independent variables (think of this matrix as a scaling factor akin to the one chosen in equation (2) to ensure an up-to-scale unique solution).

Question. Consider two different sets of parameters, $\{Parameters1\}$ and $\{Parameters2\}$, explicitly defined in the attached Maple file. Both of them satisfy the parameter constraints in (4), but under $\{Parameters1\}$ the spectral radius (ρ) of $(M + Q)$ is smaller than one ($\rho(M + Q) < 1$) and equation (6) can be rewritten as Neumann series, whereas under $\{Parameters2\}$ we have $\rho(M + Q) > 1$ and the power series decomposition of equation (6) is not possible. Also, if we apply an iterative updating process to $y_{2,i}$'s and $y_{4,i}$ from equations (2) and (3) under $\{Parameters1\}$, the solutions converge to the "correct" (up-to-scale unique) solution, whereas if we do the exact same iterative updating under $\{Parameters2\}$, the solutions diverge (see my previous post on MaplePrimes). **Importantly**, for both sets of parameters, there exist an (up-to-scale) unique solution of the linear system, which is also identical to the matrix solutions obtained in equations (5) and (6) – it's just that, under $\{Parameters1\}$, this solution can also be obtained iteratively (either by iterative 'updating' of the original system of linear equations or Neumann series decomposition), while under $\{Parameters2\}$ this solution cannot be obtained iteratively. This makes me wonder whether the equilibrium under $\{Parameters2\}$ is unstable and, if so, how to formally verify this?