Conjectures:

$$natlq_2 = \mu_{jk} + \lambda_{jk1} \cdot \mathbf{y}_1 + \lambda_{jk2} \cdot \mathbf{y}_2 + \lambda_{jk3} \cdot \mathbf{y}_3$$
$$natlq_3 = \mu_{ki} + \lambda_{ki1} \cdot \mathbf{y}_1 + \lambda_{ki2} \cdot \mathbf{y}_2 + \lambda_{ki3} \cdot \mathbf{y}_3 \tag{1}$$

Assumptions:

$$natlq_2 = \mathrm{E}\left\{X_2 \,\middle|\, Y_1,\, Y_2,\, Y_3\right\}$$
$$natlq_3 = \mathrm{E}\left\{X_3 \,\middle|\, Y_1,\, Y_2,\, Y_3\right\} \tag{2}$$

Let $\mathbf{Z} = (X_1, X_2, X_3, Y_1, Y_2, Y_3)'$ be a collection of random variables. Assume $X_1 = X_2 + X_3$ and X_2 and X_3 to be normally distributed: $X_2 \sim \mathcal{N}\left(0, \Sigma\right); \quad X_3 \sim \mathcal{N}\left(0, \Sigma\right)$. Then:

$$Y_1 \sim \mathcal{N}\left(\alpha_{ji},\ \ \beta^2_{ji2} \cdot \Sigma + \beta^2_{ji3} \cdot \Sigma + \sigma^2\right)$$
$$Y_2 \sim \mathcal{N}\left(\alpha_{jk},\ \ \beta^2_{jk2} \cdot \Sigma + \beta^2_{jk3} \cdot \Sigma + \sigma^2\right) \tag{3}$$
$$Y_3 \sim \mathcal{N}\left(\alpha_{ki},\ \ \beta^2_{ki2} \cdot \Sigma + \beta^2_{ki3} \cdot \Sigma + \sigma^2\right)$$

The collection $\mathbf{Z}$ of normal random variables can be partitioned into two sub-collections $\mathbf{Z}_1$ and $\mathbf{Z}_2$:

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}; \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}; \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}; \tag{4}$$

$$\mathbf{Z}_1 = (X_1, X_2, X_3)'; \qquad \mathbf{Z}_2 = (Y_1, Y_2, Y_3)'; \qquad \mu_1 = (\mu_{X_1}, \mu_{X_2}, \mu_{X_3})'; \qquad \mu_2 = (\mu_{Y_1}, \mu_{Y_2}, \mu_{Y_3})';$$

$$\Sigma_{11} = \begin{pmatrix} \mathrm{Var}[X_1] & \sigma_{X_1X_2} & \sigma_{X_1X_3} \\ \sigma_{X_2X_1} & \mathrm{Var}[X_2] & \sigma_{X_2X_3} \\ \sigma_{X_3X_1} & \sigma_{X_3X_2} & \mathrm{Var}[X_3] \end{pmatrix}; \qquad \Sigma_{22} = \begin{pmatrix} \mathrm{Var}[Y_1] & \sigma_{Y_1Y_2} & \sigma_{Y_1Y_3} \\ \sigma_{Y_2Y_1} & \mathrm{Var}[Y_2] & \sigma_{Y_2Y_3} \\ \sigma_{Y_3Y_1} & \sigma_{Y_3Y_2} & \mathrm{Var}[Y_3] \end{pmatrix};$$

$$\Sigma_{12} = \Sigma'_{21} = \begin{pmatrix} \sigma_{X_1Y_1} & \sigma_{X_1Y_2} & \sigma_{X_1Y_3} \\ \sigma_{X_2Y_1} & \sigma_{X_2Y_2} & \sigma_{X_2Y_3} \\ \sigma_{X_3Y_1} & \sigma_{X_3Y_2} & \sigma_{X_3Y_3} \end{pmatrix}$$

Then, if $\mathbf{Z} \sim \mathcal{N}_6(\mu, \Sigma)$, the marginal distributions of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $\mathbf{Z}_1 \sim \mathcal{N}_3(\mu_1, \Sigma_{11})$ and $\mathbf{Z}_2 \sim \mathcal{N}_3(\mu_2, \Sigma_{22})$. Therefore, for the multivariate normal distribution the overall best predictor is, in fact, a linear predictor:

$$natlq_2^* = \mu_{X_2} + (\sigma_{X_2Y_1}, \sigma_{X_2Y_2}, \sigma_{X_2Y_3}) \begin{pmatrix} \text{Var}[Y_1] & \sigma_{Y_1Y_2} & \sigma_{Y_1Y_3} \\ \sigma_{Y_2Y_1} & \text{Var}[Y_2] & \sigma_{Y_2Y_3} \\ \sigma_{Y_3Y_1} & \sigma_{Y_3Y_2} & \text{Var}[Y_3] \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{y}_1 - \mu_{Y_1} \\ \mathbf{y}_2 - \mu_{Y_2} \\ \mathbf{y}_3 - \mu_{Y_3} \end{pmatrix}$$

$$natlq_3^* = \mu_{X_3} + (\sigma_{X_3Y_1}, \sigma_{X_3Y_2}, \sigma_{X_3Y_3}) \begin{pmatrix} \text{Var}[Y_1] & \sigma_{Y_1Y_2} & \sigma_{Y_1Y_3} \\ \sigma_{Y_2Y_1} & \text{Var}[Y_2] & \sigma_{Y_2Y_3} \\ \sigma_{Y_3Y_1} & \sigma_{Y_3Y_2} & \text{Var}[Y_3] \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{y}_1 - \mu_{Y_1} \\ \mathbf{y}_2 - \mu_{Y_2} \\ \mathbf{y}_3 - \mu_{Y_3} \end{pmatrix} \tag{5}$$

Where all the covariances are endogenously determined:

$$\sigma_{Y_1Y_2} = \beta_{ji2}\beta_{jk2} \cdot \Sigma + \beta_{ji3}\beta_{jk3} \cdot \Sigma; \;\; \sigma_{Y_1Y_3} = \beta_{ji2}\beta_{ki2} \cdot \Sigma + \beta_{ji3}\beta_{ki3} \cdot \Sigma; \;\; \sigma_{Y_2Y_3} = \beta_{jk2}\beta_{ki2} \cdot \Sigma + \beta_{jk3}\beta_{ki3} \cdot \Sigma;$$
$$\sigma_{X_2Y_1} = \beta_{ji2} \cdot \Sigma; \;\; \sigma_{X_2Y_2} = \beta_{jk2} \cdot \Sigma; \;\; \sigma_{X_2Y_3} = \beta_{ki2} \cdot \Sigma; \;\; \sigma_{X_3Y_1} = \beta_{ji3} \cdot \Sigma; \;\; \sigma_{X_3Y_2} = \beta_{jk3} \cdot \Sigma; \;\; \sigma_{X_3Y_3} = \beta_{ki3} \cdot \Sigma; \tag{6}$$

To solve the problem, I match equation (5) with equation (1) and find the equilibrium values for the λs and μs. To fully characterise the equilibrium, I plug the λs and μs back into the formulas for αs and βs and obtain their equilibrium values. The model is solved.