APPM 2360 Sample Project
Chaos in Nonlinear Oscillators

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1 Introduction

You have previously studied the classic harmonic oscillator equation

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{1}$$

Also known as a mass-spring system, (1) is characterized by its response to any displacement from its equilibrium position. That is, whenever the system is displaced from equilibrium it experiences a *restoring force* that is proportional to the displacement.

The simple harmonic oscillator equation

$$m\ddot{x} + kx = 0 \tag{2}$$

is derived starting from an energy potential of the form

$$v = \frac{1}{2}kx^2$$

Under differentiation, this potential energy gives rise to a restoring force following Hooke's Law

$$F(x) = -\frac{dv}{dx} = -kx \tag{3}$$

Recall that Newton's Second Law gives the relation

$$F = ma$$

which can be expressed by a system of first order differential equations

$$\frac{dx}{dt} = v, \quad m\frac{dv}{dt} = F$$

Under the variable transformation

$$\frac{dx}{dt} = v \iff \frac{d^2x}{dt^2} = \frac{dv}{dt}$$

this system can be converted back to a single second order differential equation of the form (2).

Extending the simple harmonic oscillator equation to include an external damping force (such as friction) and a driving force gives rise to the general form of the harmonic oscillator, (1).

2 Derivation of the Duffing Equation

We will extend this idea of an oscillatory system characterized by a restoring force to problems with more complicated potential functions. Consider a more complicated energy potential of the form

$$v(x) = \hat{\alpha}\frac{x^2}{2} + \hat{\beta}\frac{x^4}{4} \tag{4}$$

Under differentiation, this potential energy gives rise to a force

$$F(x) = -\frac{dv}{dx} = -\hat{\alpha}x - \hat{\beta}x^3 \tag{5}$$

Using Newton's Second Law and the variable transformation $\frac{dx}{dt} = v \iff \frac{d^2x}{dt^2} = \frac{dv}{dt}$ as in the introduction, this system can be converted back to the single differential equation

$$m\frac{dv}{dt} = F \iff m\ddot{x} = -\hat{\alpha}x - \hat{\beta}x^3 \iff m\ddot{x} + \hat{\alpha}x + \hat{\beta}x^3 = 0$$

This gives rise to a nonlinear oscillator with a *cubic* restoring force:

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{6}$$

The equation (6) is a special case of the *Duffing Equation*, which is an ordinary differential oscillator equation whose restoring force (i.e. spring constant or stiffness) does not exactly obey Hooke's Law.

The dynamics of this equation are representative of the motion of a classical particle in a double well energy potential. We will start from this model to investigate the nonlinear oscillator equation.

Section 2: Questions

1. Plot the energy potential (4) used to derive the special case of the Duffing. To do this, let $\alpha = -1$ and $\beta = 1$ and let $x \in [-2, 2]$. Find the position and values of the energy minima of the potential function.

2. (a) Decompose the special case of the Duffing Equation (6) into a system of two first order equations $\frac{dx}{dt}$ and $\frac{dv}{dt}$.
   (b) Analytically find the nullclines and the equilibrium solutions.
   (c) Use `ode45` to solve the system over the interval $t \in [0, 10]$ starting from the initial conditions $x(0) = 1$, $v(0) = 1$.
      i. Plot both component curve solutions
      ii. Plot the phase space solution in the xv-plane. On this same plot, plot the nullclines and equilibrium solutions and the phase-portrait.

   Discuss the plots. Based on the phase-portrait, what does the stability of the equilibrium solutions appear to be?

3 Transition to Chaos: Forcing and Damping

In the section above you should have obtained a single orbit in the phase-plane solution of (6). We would say that the phase-space trajectory "closes on itself" due to energy conservation. (That is, the energy equation $V(x)$ is a Hamiltonian). In order to see the oscillator exhibit chaos, we must remove energy conservation from the system. This is done by including terms that represent both damping of the system (e.g. friction) and an external driving force. The full Duffing equation takes on the general form

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{7}$$

Section 3: Questions

1. Fully classify the Duffing Equation (7) and assign physical meaning to all of the terms this equation.

2. Rewrite the Duffing Equation (7) as a system of first order differential equations.

3. In the question above, you should have realized that the parameter γ represents the strength of the driving force in the system. Set $\delta = 0.1$, $\alpha = -1$, $\beta = 1$, and $\omega = 1.4$. Use the system found in question 2, the Matlab solver `ode45`, and the initial condition $x(0) = v(0) = 0$ to solve (7) over the following t-intervals (with a step size of $\Delta t = 0.1$) for the corresponding γ values then generate the specified plots:

   - Let $\gamma = 0.1$. Solve (7) over the interval $t \in [0, 200]$. In one figure plot the solutions over $t \in [0, 200]$ and in a second figure plot the solutions over $t \in [150, 200]$.
   - Let $\gamma = 0.318$. Solve (7) over the interval $t \in [0, 800]$. In one figure plot the solutions over $t \in [0, 200]$ and in a second figure plot the solutions over $t \in [789.85, 799]$.
   - Let $\gamma = 0.338$. Solve (7) over the interval $t \in [0, 2000]$. In one figure plot the solutions over $t \in [0, 200]$ and in a second figure plot the solutions over $t \in [1981.97, 200]$.
   - Let $\gamma = 0.35$. Solve (7) over the interval $t \in [0, 3000]$. In one figure plot the solutions over $t \in [0, 300]$, in a second figure use the command `subplot` to plot the solution over $t \in [2959.85, 3000]$ and to plot the solution over $t \in [2989.85, 3000]$.

   Discuss each of the plots. In particular,

   - Compare the beginning and ending behaviors of the phase-plane solutions. How does increasing γ influence the end behaviors? Why might this be?
   - How does changing the magnitude of γ influence the system's transition to chaos?

4 Project Guidelines

Be sure to read all guidelines very carefully!