Solving the ODE with Boundary Conditions at Infinity

1 Problem Definition

We are given the following second-order ODE:

$$\frac{d^2y(x)}{dx^2} + \frac{2}{x}\frac{dy(x)}{dx} - y(x) = 0 \tag{1}$$

with the boundary conditions:

$$y(-1) = 0, \quad \frac{dy}{dx}(-\infty) = -\frac{1}{e} \tag{2}$$

2 Solving the ODE Analytically

The ODE can be rewritten as:

$$x^2\frac{d^2y(x)}{dx^2} + 2x\frac{dy(x)}{dx} - x^2y(x) = 0 \tag{3}$$

This is a Bessel equation of order 0. The general solution is given by:

$$y(x) = C_1J_0(x) + C_2Y_0(x) \tag{4}$$

where $J_0(x)$ and $Y_0(x)$ are the Bessel functions of the first and second kind, respectively, and C_1 and C_2 are constants to be determined from the boundary conditions.

3 Applying Boundary Conditions

3.1 Boundary Condition 1: $y(-1) = 0$

At $x = -1$, we have:

$$y(-1) = 0 = C_1J_0(-1) + C_2Y_0(-1) \tag{5}$$

This gives a linear equation relating C_1 and C_2. The values of $J_0(-1)$ and $Y_0(-1)$ can be looked up in tables or computed numerically:

$$J_0(-1), \quad Y_0(-1)$$

3.2 Boundary Condition 2: $\frac{dy}{dx}(-\infty) = -\frac{1}{e}$

For large negative x, the asymptotic behavior of the Bessel functions is:

$$J_0(x) \sim \sqrt{\frac{2}{\pi|x|}} \cos\left(|x| - \frac{\pi}{4}\right)$$

$$Y_0(x) \sim \sqrt{\frac{2}{\pi|x|}} \sin\left(|x| - \frac{\pi}{4}\right)$$

Using these asymptotic expansions, we can compute the derivative of the solution $y(x)$ as $x \to -\infty$, and match it to the boundary condition:

$$\frac{dy}{dx}(-\infty) = -\frac{1}{e}$$

This provides a second equation involving C_1 and C_2.

4 Solving for the Constants C_1 and C_2

We now have two linear equations from the boundary conditions:

$$C_1 J_0(-1) + C_2 Y_0(-1) = 0$$

$$\text{Asymptotic behavior at } x \to -\infty \text{gives another equation.}$$

By solving these two equations, we can find the constants C_1 and C_2.

5 Numerical Solution Approach

If solving analytically becomes difficult due to the complexity of boundary conditions, a numerical solution can be used instead.

The ODE can be solved numerically using a numerical approximation for the boundary at $x = -\infty$, say $x = -100$. In Maple, the following code can be used to solve the ODE numerically:

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ode := diff(diff(y(x), x), x) + 2*diff(y(x), x)/x - y(x) = 0:
IC := y(-1)=0, D(y)(-100)=-1/exp(1):
numeric_sol := dsolve([ode, IC], y(x), numeric):
plots:-odeplot(numeric_sol, [x, y(x)], x=-100..-1);
```

This will give a numerical solution that approximates the behavior of the function and its derivative near $x = -\infty$.

6 Conclusion

The solution to the ODE can be found analytically using Bessel functions, with the constants C_1 and C_2 determined by the boundary conditions. Alternatively, a numerical solution can be used when handling boundary conditions at infinity.