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HEAT TRANSFER IN POROUS MEDIA WITH PHASE CHANGE*

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Abstract—Liquid from a constant temperature reservoir enters a porous medium. At the opposite end the temperature of the system is raised above the evaporation point of the liquid by imposing either a constant temperature, or a constant heat flux. This causes an evaporation front to propagate into the material, separating a vapor region from the liquid region. The object of the paper is to analyze this system, investigating the relative importance of convection versus conduction, and the main parameters influencing the temperature distribution, and the interface position. Exact solutions are obtained for the steady-state problem where (1) properties are constant, and (2) density and viscosity of the vapor are variable. Also, an approximate solution of the constant property, transient problem is obtained by applying a heat-balance integral technique.

NOMENCLATURE

C_p,	specific heat;
d,	average pore diameter;
g,	acceleration of gravity;
h,	enthalpy;
h_{fg},	latent heat of vaporization;
k,	thermal conductivity;
K,	permeability;
L,	length of porous section;
$\dot{m}$,	mass flow rate per unit area $= \rho v$;
P,	pressure;
Pe,	Péclét number $= \dot{m}C_pL/k = RePrL/d$;
Pr,	Prandtl number $= \mu C_p/k$;
q,	heat flux per unit area;
Ra,	Rayleigh number $= gLK\beta\,\Delta T/\alpha\nu$;
Re,	Reynolds number $= \rho vd/\mu$;
S,	interface position;
t,	time;
T,	temperature;
u,	actual velocity in pores;
v,	Darcy velocity;
x,	distance coordinate;
α,	thermal diffusivity;
β,	coefficient of thermal expansion;
ε,	porosity;
η,	transformed coordinate $= x - S/1 - S$;
θ,	dimensionless temperature;
μ,	viscosity;
ν,	kinematic viscosity $= \mu/\rho$;
ξ,	transformed coordinate $= S - x/S$;
ρ,	density.

Dimensionless quantities

C,	C_{pL}/C_{pV};
E,	$(\rho C_p)_{L_{eff}}/\rho_L C_{pL}$;
F,	$\dot{m}_L/\dot{m}_i$;
H,	$h_{fg}(T^*)/[h_L(T^*) - h_L(T_R)]$;
R,	ν_V/ν_L;
$\hat{t}$,	$t/(L^2\rho_L C_{pL}/k_{L_{eff}})$;
V,	$(\rho C_p)_{V_{eff}}/\rho_v C_{pV}$;
Y,	ρ_L/ρ_V;
κ,	$k_{L_{eff}}/k_{V_{eff}}$.

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Subscripts

eff, effective property;
sat, saturated porous medium;
ss, steady state;
i, initial value (before interface advances into medium);
R, reservoir ($x = L$);
f, fluid (liquid or vapor);
L, liquid;
V, vapor;
S, solid.

Superscripts

$*$ saturation conditions;
$\sim$ average quantity;
$\hat{x}$, dimensionless quantity.

INTRODUCTION

MOST of the previous work on heat transfer in porous media deals with a single phase, either liquid or gas, flowing through the material. However, in certain problems involving transpiration cooling and heat shields there may be regions of different phases separated by a moving phase-change interface. In the problem discussed in [1] and [2] liquid from a constant temperature reservoir flows into one end of a porous medium. At the opposite end of the either a surface heat flux or a constant temperature, where the maximum temperature is less than the boiling point of the liquid. If this restriction is relaxed, an evaporation front, separating a vapor region from the liquid region, may propagate into the medium. The object of this paper is to analyze this system, showing the relative importance of convection versus conduction, and the main parameters influencing the temperature distribution, and interface position. The material is treated as a continuum, neglecting the pore structure, in order to apply "average" governing equations. The one-dimensional problem is solved neglecting radiation and natural convection and assuming a small, appropriately defined, Reynolds number so that Darcy's law may be applied. Exact solutions of the steady-state problem are obtained for (1) constant properties, and (2) variable density and viscosity of the vapor. An approximate solution of the transient problem with constant properties is obtained using the heat-balance integral technique proposed by Goodman [3, 4] and applied successfully in [5–7].

For the case with temperature prescribed on the boundary, we find one steady-state interface position which propagates farther into the

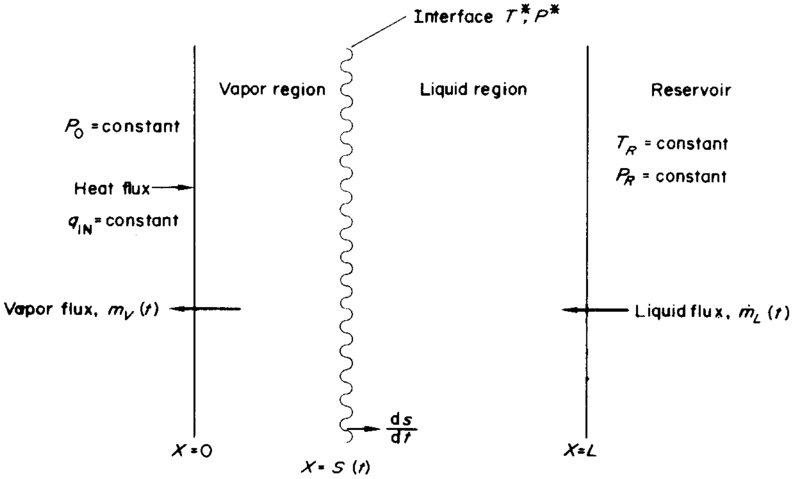

FIG. 1. Diagram of physical model.

medium as the boundary temperature is increased. For constant surface heat flux there can be either one or three steady-state interface positions depending on the parameters of the problem. A stability analysis shows that for the later case the middle position is unstable. The solution of the transient problem shows the initial conditions determining which stable, steady-state interface position is reached.

FORMULATION OF THE PROBLEM

A schematic diagram of the problem is presented in Fig. 1. Liquid, at temperature T_R and pressure P_R, is forced through a porous medium by an imposed pressure gradient, $P_R > P_0$. The liquid is assumed to flow from an infinite reservoir at $x = L$ so that T_R and P_R may be assumed constant. It is assumed that the mass is removed instantly at $x = 0$, and the pressure P_0 is constant. Also, the pressure difference due to surface tension effects across the liquid–vapor interface is assumed small compared to $(P_R - P_0)$ so that $P(x = S^+) = P(x = S^-)$. For a porous structure with a small radius of curvature this discontinuity in pressure may have to be included. Radiation and natural convection are neglected compared to conduction and forced convection. Natural convection can be neglected if an appropriately defined Rayleigh number, $Ra = gLK\beta\,\Delta T/\alpha_{sat}\nu_f$, *is small.*†

If a constant temperature, T_0, greater than the saturation temperature of the liquid at pressure P_0, is applied at $x = 0$, an evaporation interface will propagate into the medium, separating a vapor region from the liquid region. If a sufficiently high constant heat flux per unit area, q_{in}, is applied at $x = 0$ in the direction opposite the mass flux, an evaporation interface will also propagate into the medium. For both of these cases, the temperature and pressure at the interface are the saturation temperature and pressure, T^* and P^* respectively, where $P_0 \leqslant P^* \leqslant P_R$. It is assumed that the maximum temperature in the material is less than the melting temperature of the solid. For simplicity the one-dimensional problem is treated.

Because of the complex geometry it is impossible to formulate the problem in terms of the actual fluid flow in the pores. As in most studies of flow through porous media, the heterogeneous solid–fluid system is treated as a continuum, which allows average or "macroscopic" governing equations to be applied.

The equation of flow known as Darcy's law,

$$v = -\frac{K}{\mu}\frac{\mathrm{d}P}{\mathrm{d}x} \tag{1}$$

was first deduced experimentally in 1856 [10]. Since then there have been several attempts to derive equation (1) analytically starting with the Navier–Stokes equations [11–13]. These "derivations" show that Darcy's law is restricted to flow in which viscous forces dominate over inertia forces. We assume a small, appropriately defined Reynolds number,

$$Re \equiv |\rho v d/\mu| < 1 \tag{2}$$

so that equation (1) may be applicable.

For the transient problem, a modified Darcy's law is sometimes used [9, 14] in the form

$$\frac{\partial v}{\partial t} + \frac{\nu}{K}v = -\frac{1}{\rho}\frac{\partial P}{\partial x}.$$

If the viscous time K/ν is small, except for the first fraction of a second, the term $\partial v/\partial t$ may be neglected.† Although Darcy's law was determined empirically with constant viscosity, it is often assumed valid for variable viscosity as well. Wooding [15, 16] uses equation (1) with temperature dependent viscosity, and Dybbs

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† According to Scheidegger [8], for convection currents to develop in porous media a temperature gradient greater than $4\pi^2\alpha_{sat}\,\nu f/L^2Kg\beta$ is necessary. Experiments by Combarnous and LeFur [9] show that for $Ra < 40$ natural convection is negligible. In the examples presented in this paper $Ra < 1$.

† For example, a typical value for K/ν for water flowing through packed sand is about 10^{-5} s.

[13] shows analytically that equation (1) may not be restricted to constant viscosity.

The mass conservation equation may be written in the form [8, 9]:

$$\varepsilon\frac{\partial\rho_f}{\partial t}+\frac{\partial}{\partial x}(\rho_f v)=0. \tag{3}$$

To simplify the problem, and consistent with a small Reynolds number, it is assumed the flow is slow enough so the temperature of the solid and the adjacent fluid are equal. This allows us to treat the temperature field as continuous. In order to apply an average energy equation, it is necessary to determine an effective thermal conductivity of the saturated porous medium. Hashin and Shtrikman [17] derive an equation for the upper and lower bounds of the effective conductivity of heterogeneous materials, and [18–24] discuss the prediction of effective conductivities. For the problem under consideration it is assumed that the effective conductivity is known, at least within reasonable limits, and taking an upper and lower bound will yield limits on the solution of the problem. Hence, the average energy equation can be written in the form [9, 25]:

$$(\rho C_p)_{\mathrm{eff}}\frac{\partial T}{\partial t}+\rho_f C_{p_f} v\frac{\partial T}{\partial x}=\frac{\partial}{\partial x}\left(k_{\mathrm{eff}}\frac{\partial T}{\partial x}\right) \tag{4}$$

where,

$$(\rho C_p)_{\mathrm{eff}}=(1-\varepsilon)\,\rho_s C_{ps}+\varepsilon\rho_f C_{pf}.$$

consisting with assuming same fluid and solid temperatures.

The solution of the temperature distribution and interface position involves the simultaneous solution of equations (1), (3) and (4) in both the liquid and vapor regions subject to appropriate boundary conditions at $x = 0$, $x = L$, and at the interface.

BOUNDARY CONDITIONS

The boundary conditions applicable to this problem are:

at $x = 0$: $P_v = P_0$,

$$\begin{cases}\text{case (a): } T_v = T_0 & (5)\\[4pt] \text{case } (b)\text{: } q_{\mathrm{in}} = -k_{\mathrm{eff}}\dfrac{\partial T_v}{\partial x} & (6)\end{cases}$$

$$\text{at } x = L\text{: } P_L = P_R,\quad T_L = T_R \tag{7}$$

at the interface,

$$x = S(t)\text{: } P_v = P_L = P^*,\ T_v = T_L = T^*. \tag{8}$$

The physical properties at the interface are related through the equation of state,

$$T^* = T^*(P^*). \tag{9}$$

It can be shown that under certain conditions equation (8) can be simplified to $T^* =$ constant. Since $P_0 \leqslant P^* \leqslant P_R$, we obtain a restriction on the saturation temperature:

$$T^*(P_0) \leqslant T^*(P^*) \leqslant T^*(P_R). \tag{10}$$

For the case where $T^*(P_0) \approx T^*(P_R)$ we can assume that the saturation temperature is known. The conditions when this occurs is presented below. Substituting equation (1) into (2) we obtain a restriction on the pressure gradient,

$$\frac{\mathrm{d}P}{\mathrm{d}x} < \left|\frac{\rho_f v_f{}^2}{Kd}\right|. \tag{11}$$

When the right hand side of inequality (10) is small, the difference between P_R and P_0 is small, and for many fluids $T^*(P_0) \approx T^*(P_R)$.† We will restrict the problem to these cases and assume T^* is a known constant.

$$T^* = T^*\left(\frac{P_R - P_0}{2}\right). \tag{12}$$

The equations in the liquid and vapor region are coupled by a mass balance and an interface

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† As an example, for water flowing through packed sand $(\rho_j v_j/K\mathrm{d}) = 0\ (10^4\ \mathrm{N}/m^3)$. In a porous bed of length up to 1m the pressure difference $P_R - P_0$ is less $10^3\ \mathrm{N}/m^2$ For this pressure difference the saturation temperature of water around a pressure of one atmosphere varies by less than 2°C [26].

energy equation. Here it is necessary to point out the difference between two different average velocities: the Darcy velocity in equation (1), and the average fluid velocity in the pores. The Darcy velocity is a velocity which when multiplied by the fluid density and the total cross sectional area of the porous medium yields the mass flux of the fluid. By definition,

$$v \equiv \frac{1}{A_{\text{total}}} \int_{A_{\text{total}}} u \, \mathrm{d}A \tag{13}$$

where $A_{\text{total}} = A_{\text{voids}} + A_{\text{solid}}$. Since u is zero on the solid, the above integration is carried out over the void cross sectional area only. The secondary velocity, u, is the average fluid velocity in the voids. This velocity, when multiplied by the fluid density and the void cross sectional area only, yields the mass flux. Therefore,

$$\tilde{u} \equiv \frac{1}{A_{\text{voids}}} \int_{A_{\text{voids}}} u \, \mathrm{d}A. \tag{14}$$

Combining equations (13) and (14), and introducing,

$$A_{\text{voids}} = \varepsilon A_{\text{total}} \tag{15}$$

yields,

$$v = \varepsilon \tilde{u}. \tag{16}$$

The significance of the average velocity in the pores becomes apparent when considering a mass balance across a moving interface. Since the actual flow is only in the pores, the total liquid mass flux per unit time through the moving phase change interface equals

$$\rho_L \left(\frac{\mathrm{d}S}{\mathrm{d}t} - \tilde{u}_L \right) A_{\text{voids}}.$$

This must equal the vapor flux leaving the interface.

$$\rho_L \left(\frac{\mathrm{d}S}{\mathrm{d}t} - \tilde{u}_L \right) A_{\text{voids}} = \rho_V \left(\frac{\mathrm{d}S}{\mathrm{d}t} - \tilde{u}_V \right) A_{\text{voids}}.$$

Substituting equations (15) and (16) into the above equation and dividing by A_{total} yields,

$$\rho_L \left(\varepsilon \frac{\mathrm{d}S}{\mathrm{d}t} - v_L \right) = \rho_V \left(\varepsilon \frac{\mathrm{d}S}{\mathrm{d}t} - v_V \right). \tag{17}$$

Rearranging (17) we obtain the mass balance,

$$\dot{m}_V(t) = \dot{m}_L(t) - \varepsilon(\rho_L - \rho_V) \frac{\mathrm{d}S}{\mathrm{d}t} \tag{18}$$

where $\dot{m}$ is the mass flux per unit area. Equation (18) states that the vapor flux leaving the porous medium at $x = 0$ equals the liquid flux entering at $x = L$ plus the rate of decrease of mass in the system.

The interface energy equation comes from an energy balance at $x = S(t)$. Consider the energy convected into and out of the moving interface

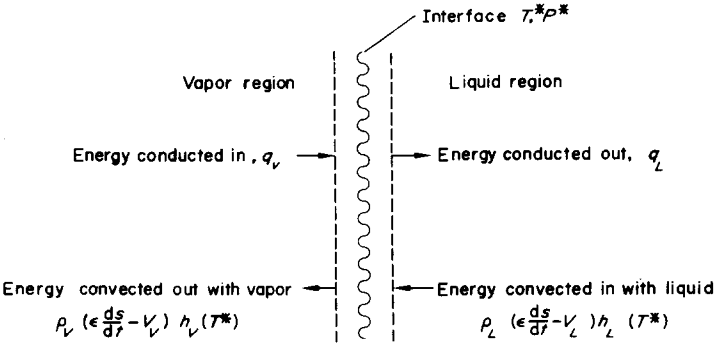

FIG. 2. Diagram of energy balance at the vaporizing interface.

with the fluid. Making the energy balance as shown in Fig. 2 yields,

$$q_L + \rho_L\left(\varepsilon\frac{\mathrm{d}S}{\mathrm{d}t} - v_L\right)h_L(T^*)$$

$$- q_V - \rho_V\left(\varepsilon\frac{\mathrm{d}S}{\mathrm{d}t} - v_V\right)h_V(T^*) = 0.$$

Substituting equation (17) and the definition of the latent heat of vaporization, $h_{fg}(T^*) = h_V(T^*) - h_L(T^*)$, into the above relation yields,

$$\text{at } x = S(t)\text{: } k_{L_{\mathrm{eff}}}\frac{\partial T_L}{\partial x} - k_{V_{\mathrm{eff}}}\frac{\partial T_V}{\partial x}$$

$$= \rho_L\left(\varepsilon\frac{\mathrm{d}S}{\mathrm{d}t} - v_L\right)h_{fg}(T^*) \qquad (19)$$

where from Fourier's law of heat conduction, $q = -k(\mathrm{d}T/\mathrm{d}x)$. Equation (19) states that the difference between the heat conducted into and out of the phase change interface equals the energy required to vaporize the liquid.

We now have the governing equations (1), (3) and (4) in the liquid and vapor regions subject to conditions (5)–(7), (12), (18) and (19). An exact solution of the steady-state problem and an approximate solution of the transient problem will be obtained for constant properties.

ANALYSIS AND RESULTS

Steady-state—constant properties

In the constant property case it is possible to solve for the mass flux as a function of the interface position. Substituting equation (1) into (3) yields,

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 0. \qquad (20)$$

Solving equation (20) for the pressure distribution in each region subject to boundary conditions (5)–(7), and combining the result with equation (1) yields the mass flux,

$$\dot{m}_L = -\frac{K}{v_L}\frac{(P_R - P^*)}{(L - S)} \qquad (21)$$

$$\dot{m}_V = -\frac{K}{v_V}\frac{(P^* - P_0)}{S}. \qquad (22)$$

Substituting equations (21) and (22) into the mass conservation equation (18) yields a relation for P^* in terms of S. Combining this result with (21) we obtain,

$$\dot{m}_V = \dot{m}_L = -\frac{K}{v_L}\frac{(P_R - P_0)}{(RS + 1 - S)} \qquad (23)$$

where $R \equiv v_V/v_L$.

The energy equation (4) in each region can be written in the form

$$k_{\mathrm{eff}}\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - C_{P_f}\dot{m}_f\frac{\mathrm{d}T}{\mathrm{d}x} = 0 \qquad (24)$$

where $\dot{m}_f$ is obtained from equation (23). We will solve for the temperature distribution and interface position in two cases: (a) constant temperature prescribed on the boundary $x = 0$, and (b) constant surface heat flux at $x = 0$.

(a) *Constant temperature boundary condition.* Equations (5)–(7), (19), (23) and (24) are written in nondimensional form using the following quantities:

$$\hat{S} \equiv \frac{S}{L}, \quad \hat{x} = \frac{x}{L}, \quad C \equiv \frac{C_{P_L}}{C_{PV}}, \quad \kappa \equiv \frac{k_{L_{\mathrm{eff}}}}{k_{V_{\mathrm{eff}}}},$$

$$H \equiv \frac{h_{fg}(T^*)}{h_L(T^*) - h_L(T_R)}, \quad \theta_V \equiv \frac{T_0 - T_V}{T_0 - T^*},$$

$$\theta_L \equiv \frac{T_L - T_R}{T^* - T_R}, \quad \theta_0 \equiv \frac{T_0 - T^*}{T^* - T_R}, \quad F \equiv \frac{\dot{m}_L}{\dot{m}_i}$$

$$Pe_i \equiv \frac{\dot{m}_i C_{P_L} L}{k_{L_{\mathrm{eff}}}}$$

where the initial Péclét number is based on the flow rate when the interface is at $x = 0$, $\dot{m}_i = -K(P_R - P_0)/Lv_L$. The energy equation (24) in each region becomes:

$$\frac{\mathrm{d}^2\theta_V}{\mathrm{d}x^2} + \frac{\kappa\, Pe_i F(S)}{C}\frac{\mathrm{d}\theta_V}{\mathrm{d}x} = 0, \quad 0 \leqslant x \leqslant S \qquad (25)$$

$$\frac{\mathrm{d}^2\theta_e}{\mathrm{d}x^2} \quad Pe_i F(S)\frac{\mathrm{d}\theta_L}{\mathrm{d}x} = 0, \quad S \leqslant x \leqslant 1 \qquad (26)$$

where the superscript "Ê" has been dropped, and $F(S) = 1/(RS + 1 - S)$. The normalized boundary conditions are:

$$\text{at} \quad x = 0: \quad \theta_V = 0 \tag{27}$$

$$\text{at} \quad x = 1: \quad \theta_L = 0 \tag{28}$$

$$\text{at } x = S \begin{cases} \theta_V = \theta_L = 1 & (29) \\ \dfrac{d\theta_L}{dx} + \dfrac{\theta_0}{\kappa}\dfrac{d\theta_V}{dx} = HPe_iF(S). & (30) \end{cases}$$

Equations (25) and (26) are integrated directly applying boundary conditions (27)–(29) to obtain the following expressions:

$$\theta_V(x) = \frac{1 - \exp(-\kappa Pe_iFX/C)}{1 - \exp(-\kappa Pe_iFS/C)}, \; 0 \leqslant x \leqslant S \tag{31}$$

$$\theta_L(x) = \frac{\exp[Pe_iE(1 - X)] - 1}{\exp[Pe_iF(1 - S)] - 1}, \; S \leqslant x \leqslant 1. \tag{32}$$

Substituting these results into the energy balance condition (30) yields the following transcendental equation for the position of the interface in terms of the independent parameters C, K, H, θ_0, Pe_i, and R.

$$\frac{\theta_0}{C} = \left\{\exp\left[\frac{Pe_i\,\kappa S}{C(RS + 1 - S)}\right] - 1\right\} \times \left\{H + \frac{1}{1 - \exp\left[-\dfrac{Pe_i(1 - S)}{RS + 1 - S}\right]}\right\} \tag{33}$$

It can be shown that equation (33) has only one real root for $0 \leqslant S \leqslant 1$ which, when substituted into equations (31) and (32) yields the complete solution for the temperature distribution. Asymptotic solutions for S for small and large initial Péclét numbers are:

$$\text{as } Pe_i \to 0: \mathrm{S} = \frac{\theta_0/\kappa}{1 + \theta_0/\kappa} + 0(Pe_i) \tag{33a}$$

$$\text{as } Pe_i \to \infty: S = \frac{C}{\kappa Pe_i}\ln\left[1 + \frac{\theta_0}{C(H + 1)}\right] + 0\left(\frac{1}{Pe_i^2}\right). \tag{33b}$$

Typical results are presented in Figs. 3–5.

The results in Figs. 3–10 are for water flowing

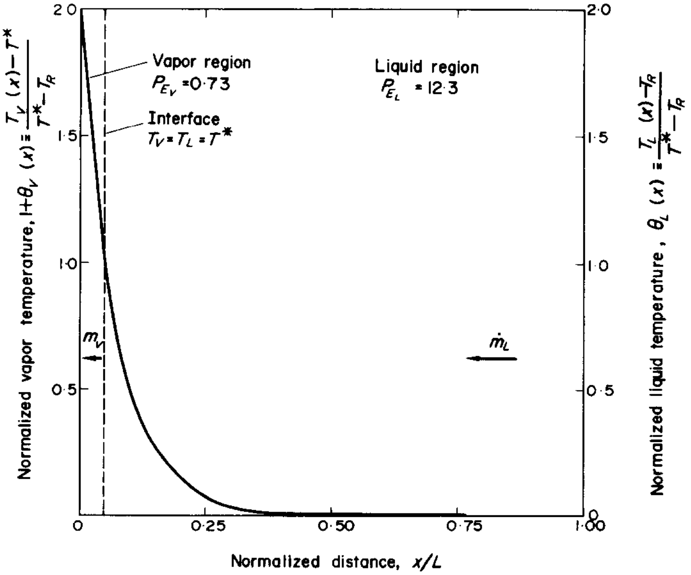

FIG. 3. Typical temperature profile for constant properties, steady-state, with surface temperature $\theta_0 = 14$, and initial Péclét number, $Pe_i = 41$.

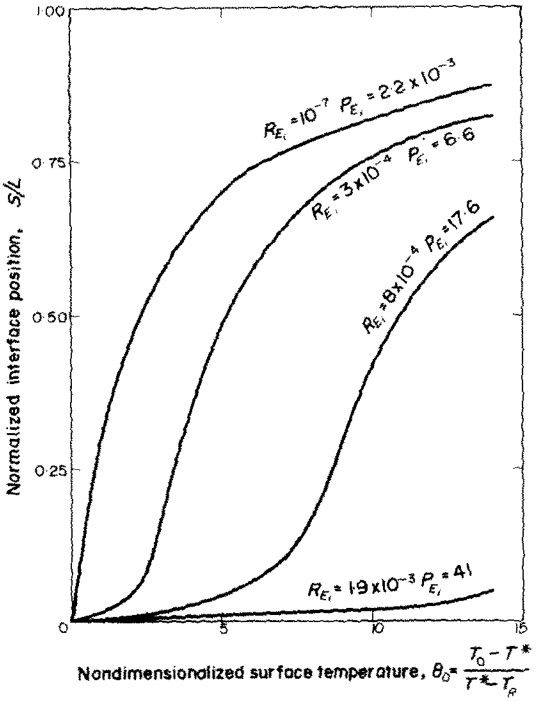

FIG. 4. Steady-state interface position, S/L, as a function of surface temperature, θ_0.

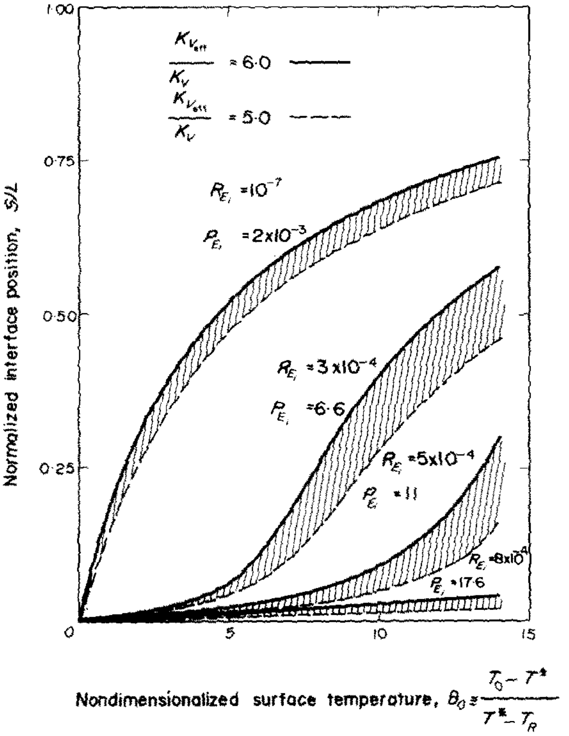

FIG. 5. Steady-state interface position as a function of surface temperature, θ_0, for a 20 per cent uncertainty in the effective thermal conductivity of the vapor region.

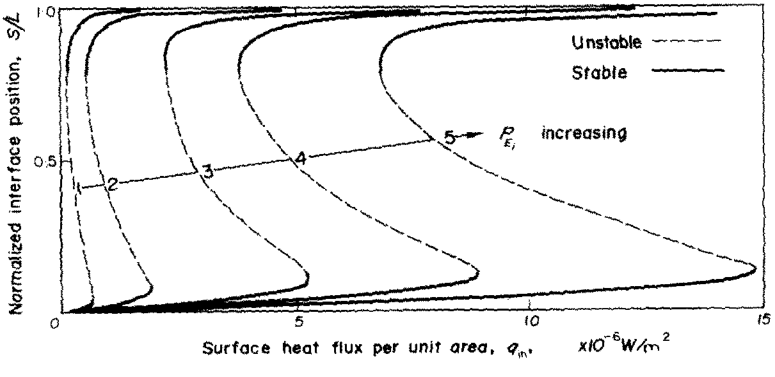

FIG. 6. Steady-state interface position, S/L, as a function of surface heat flux q_{in}, for constant properties.

1. $Re_i = 7\times10^{-3}$, $Pe_i = 77$, $q_{min} = 1·91\times10^5$ W/m^2
2. $Re_i = 9\times10^{-3}$, $Pe_i = 99$, $q_{min} = 2·46\times10^5$ W/m^2
3. $Re_i = 11\times10^{-3}$, $Pe_i = 121$, $q_{min} = 3.00\times10^5$ W/m^2
4. $Re_i = 12\times10^{-3}$, $Pe_i = 132$, $q_{min} = 3·28\times10^5$ W/m^2
5. $Re_i = 13\times10^{-3}$, $Pe_i = 143$, $q_{min} = 3·55\times10^5$ W/m^2

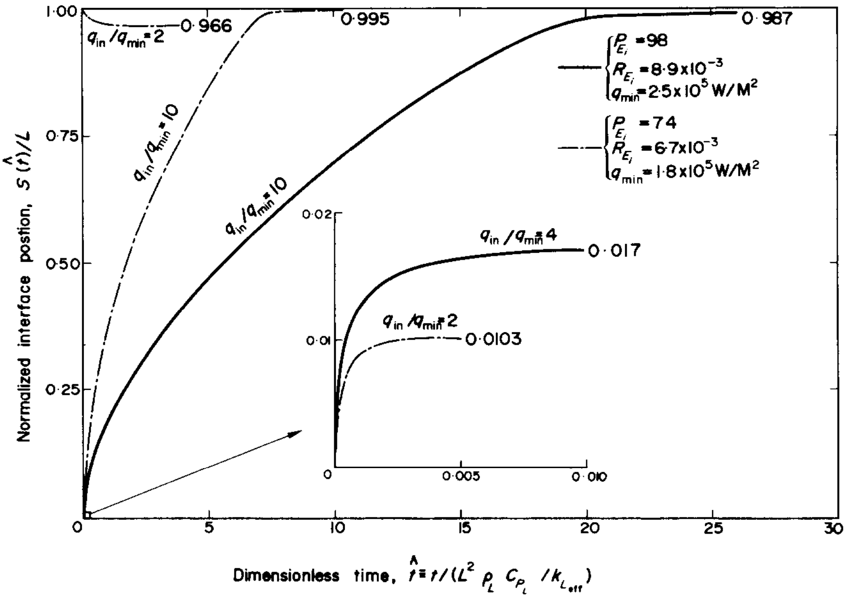

FIG. 7. Interface position, $S(\hat{t})/L$, as a function of time, for constant properties with prescribed surface heat flux, q_{in}.

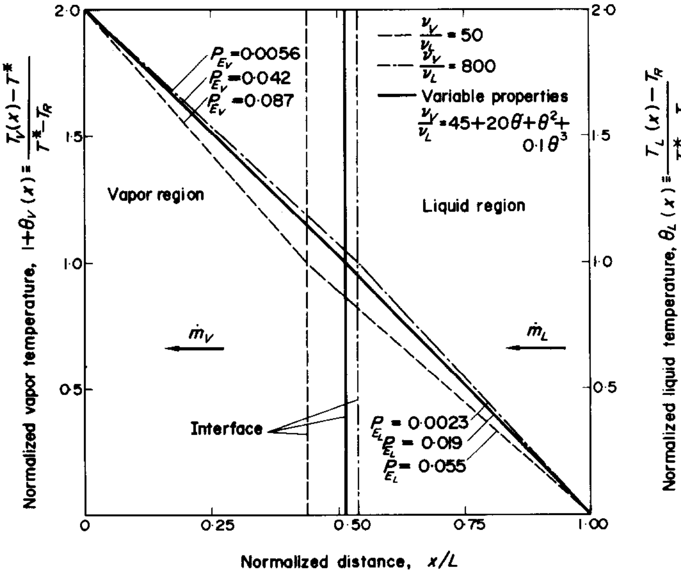

FIG. 8. Typical temperature profile comparing constant property to variable property case with surface temperature, $\theta_0 = 5$, and initial Péclét number, $Pe_i = 2·2$.

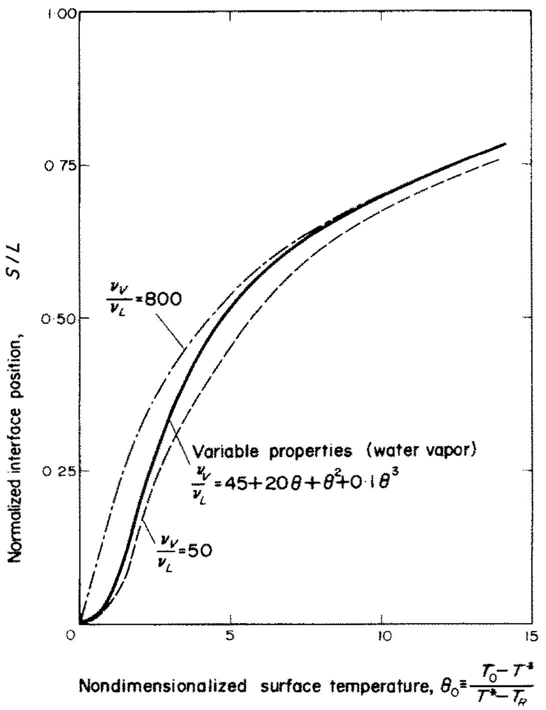

FIG. 9. Interface position, S/L, as a function of surface temperature, θ_0, for constant and variable properties with initial Péclét number, $Pe_i = 2{\cdot}2$.

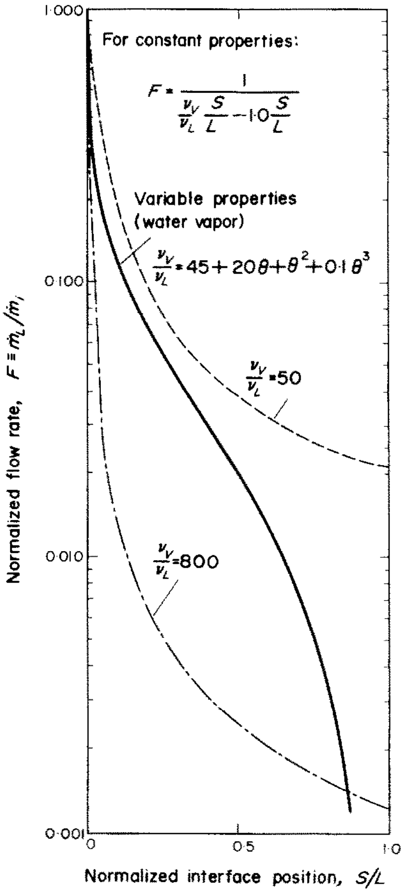

FIG. 10. Flow rate, F, as a function of interface position, S/L, for constant and variable properties with initial Péclét number, $Pe_i = 2{\cdot}2$.

through a porous bed of packed spheres. Typical values for the Prandtl number of water, the porosity, and L/d are taken as 2·2, 0·4 and 10^4 respectively. Also, $R = 45$ for the constant property cases, $H = 7{\cdot}1$, $C = 1{\cdot}96$, and $\kappa = 4{\cdot}0$ except where otherwise noted. The reservoir temperature is 25°C, P_0 is one atmosphere, and the saturation temperature is assumed to be 100°C. The properties of the solid and liquid are kept constant while the effect of varying the parameters T_0, q_{in}, Pe, k_V, $k_{V_{eff}}$ and ν_V are examined.

Figure 3 shows a typical temperature distribution in the porous medium with $\kappa = 2{\cdot}24$. In the vapor region Pe is small, conduction dominates over convection, and the temperature profile is linear. In the liquid region Pe is high, convection dominates, and the temperature distribution is exponential.

Figure 4 presents typical curves for the steady-state interface position versus the prescribed boundary temperature with variable Pe_i, and $\kappa = 2{\cdot}24$. The Péclét number and Reynolds number are related by[†] $Pe = RePr\, L/d$. The lower Pe_i the farther the interface penetrates into the material, where the limit as $Pe_i \rightarrow 0$ is given by equation (33a). For large Pe_i, convection completely dominates the flow and the interface remains near $x = 0$.

As mentioned in the section "Formulation of the Problem", it is usually possible to obtain only upper and lower bounds for the effective thermal conductivity of porous media. Figure 5 shows the effect of a 20 per cent variation between the maximum and minimum k_{eff} in the vapor region. For this example ($\kappa_{\max} = 5{\cdot}72$ and $\kappa_{\min} = 4{\cdot}77$) when $Pe = 11$ and $\theta_0 = 14$ the range of the solution for the steady-state interface position varies by more than a factor of two; but for $\theta_0 < 14$ and Pe_i greater than approximately 20 the variation of the interface position is almost negligible. In general, the accurate prediction of k_{eff} is more important for flow with convection and conduction than for either convection dominated ($Pe_i \gg 1$) or conduction dominated ($Pe_i \ll$) flow.

(b) *Constant heat flux boundary condition.* For this case it is necessary to determine the minimum surface heat flux to cause the liquid to boil at $x = 0$. This is found to be (see Appendix I),

$$q_{\min} = \frac{Pe_i k_{L_{eff}}}{LC_{P_L}}\left[h_{fg}(T^*) + \frac{h_L(T^*) - h_L(T_R)}{1 - e^{-Pe_i}}\right]. \quad (34)$$

Defining the dimensionless quantities:

$$\theta_v \equiv \frac{T_V - T_R}{T^* - T_R}, \qquad \theta_L \equiv \frac{T_L - T_R}{T^* - T_R},$$

$$Q \equiv \frac{q_{\text{in}}}{q_{\min}} > 1$$

and using the normalized quantities of case (a), we obtain the governing equations in an identical form to equations (25) and (26). The corresponding boundary conditions are:

$$\text{at} \quad x = 0: \frac{d\theta_V}{dx} = \frac{\kappa Pe_i Q(H - H\,e^{-Pe_i} + 1)}{(1 - e^{-Pe_i})} \equiv -\Gamma \quad (35)$$

$$\text{at} \quad x = 1: \theta_L = 0 \quad (36)$$

$$\text{at } x = S \quad \begin{cases} \theta_L = \theta_V = 1 & (37) \\ \dfrac{d\theta_L}{dx} - \dfrac{1}{\kappa}\dfrac{d\theta_V}{dx} = HPe_i F(S). & (38) \end{cases}$$

Integrating equations (25) and (26) subject to boundary conditions (35)–(37) we obtain the temperature distribution,

$$\theta_V(x) = 1 + \frac{CQ(H - H\,e^{-Pe_i} + 1)}{F(1 - e^{-Pe_i})} \times (e^{-\kappa Pe_i Fx/C} - e^{-\kappa Pe_i FS/C}). \quad (39)$$

$$0 \leqslant x \leqslant S$$

The solution for $\theta_L(x)$ is identical in form to equation (32). Substituting these results into boundary condition (38) yields the following transcendental equation for the interface position:

$$\left\{\frac{Q(RS + 1 - S)(H - H\,e^{-Pe_i} + 1)\exp\left[-\kappa Pe_i S/C(RS + 1 - S)\right]}{(1 - e^{-Pe_i})} - H\right\}\left\{1 - \exp\left[-\frac{Pe_i(1 - S)}{RS + 1 - S}\right]\right\} = 1. \quad (40)$$

Solving for S in terms of the independent parameters C, κ, H, Q, Pe_i and R, and substituting equations (32) and (39), yields the complete solution for the temperature distribution. Asymptotic solutions for S for small and large initial Peclét number are:

$$\text{as} \quad Pe_i \to 0: \quad S = \frac{Q - 1}{Q} + 0(Pe_i) \quad (40a)$$

$$\text{as} \quad Pe_i \to \infty: \quad S = \frac{C}{\kappa Pe_i}\ln Q + 0\left(\frac{1}{Pe_i^2}\right). \quad (40b)$$

For $Pe_i \ll 1$ and $Pe_i \gg 1$ the interface position is given by equations (40a) and (40b) respectively. Figure 6 presents typical results for S as a function of heat input for initial Péclét numbers of order 100. For Pe_i small, but larger than about one, and $q_{\mathrm{in}} > q_{\mathrm{min}}$, the interface will penetrate into the medium until it reaches a steady state at the reservoir. For large Pe_i convection dominates the flow, heat is not conducted into the system, and the interface does not penetrate. For intermediate values of the surface heat flux and Pe_i there may be three steady-state interface positions The position where two of the solutions merge is called the metastable solution. One method to find this point is shown in the next section where the stability of each of the steady-state solutions is investigated.

Stability analysis

A stability analysis involves the study of the transient governing equations (1), (3) and (4) in the liquid and vapor region subject to conditions (5b), (6), (7), (12), (18) and (19).

In the steady-state problem the mass flux of liquid and vapor is given by equations (21) and (22) respectively. For the transient case the liquid flux remains the same, but there is an additional flux $\varepsilon(\rho_L - \rho_v)(\mathrm{d}S/\mathrm{d}t)$, which is equal to the rate of decrease of mass in the system.

$$\dot{m}_V = -\frac{K}{v_V}\frac{(P^* - P_0)}{S} - \varepsilon(\rho_L - \rho_V)\frac{\mathrm{d}S}{\mathrm{d}t}. \qquad (41)$$

Combining equations (18), (21) and (41) and solving for P^* we obtain the liquid flux,

$$\dot{m}_L(t) = -\frac{K(P_R - P_0)}{v_L[RS(t) + 1 - S(t)]}. \qquad (42)$$

The vapor flux is given by equation (18). Substituting equation (18) into (4) allows us to write the energy equation as:

$$k_{v_{\mathrm{eff}}}\frac{\partial^2 T_v}{\partial x^2} + \left[\varepsilon(\rho_L - \rho_v)\frac{\mathrm{d}S}{\mathrm{d}t} - \dot{m}_L(t)\right]C_{pV}\frac{\partial T_v}{\partial x}$$
$$= (\rho C_p)_{v_{\mathrm{eff}}}\frac{\partial T_v}{\partial t}, \quad 0 \leqslant x \leqslant S \qquad (43)$$

$$k_{L_{\mathrm{eff}}}\frac{\partial^2 T_L}{\partial x^2} - \dot{m}_L(t)\,C_{p_L}\frac{\partial T_L}{\partial x}$$
$$= (\rho C_p)_{L_{\mathrm{eff}}}\frac{\partial T_L}{\partial t}, \quad S \leqslant x \leqslant L. \qquad (44)$$

Using previously defined quantities and the following:

$$Y \equiv \frac{\rho_L}{\rho_v}, \quad E \equiv \frac{(\rho C_p)_{L_{\mathrm{eff}}}}{\rho_L C_{p_L}}, \quad V \equiv \frac{(\rho C_p)_{v_{\mathrm{eff}}}}{\rho_v C_{pV}},$$
$$\hat{t} \equiv \frac{t}{L^2 \rho_L C_{p_L}/k_{L_{\mathrm{eff}}}}$$

we obtain the nondimensional energy equations,

$$\frac{\partial^2 \theta_V}{\partial x^2} + \frac{\kappa}{C}\left[\varepsilon\left(1 - \frac{1}{Y}\right)\frac{\mathrm{d}S}{\mathrm{d}\hat{t}} + Pe_i F(S)\right]\frac{\partial \theta_V}{\partial x}$$
$$= \frac{V\kappa}{CY}\frac{\partial \theta_V}{\partial \hat{t}}, \quad 0 \leqslant x \leqslant S \qquad (45)$$

$$\frac{\partial^2 \theta_L}{\partial x^2} + Pe_i F(S)\frac{\partial \theta_L}{\partial x} = E\frac{\partial \theta_L}{\partial \hat{t}}, \quad S \leqslant x \leqslant 1. \qquad (46)$$

These are subject to conditions (35)–(37), and the nondimensional energy balance condition,

$$\text{at } x = S(\hat{t}): \frac{\partial \theta_L}{\partial x} - \frac{1}{\kappa}\frac{\partial \theta_V}{\partial x}$$
$$= \left[Pe_i F(S) + \varepsilon\frac{\mathrm{d}S}{\mathrm{d}\hat{t}}\right]H. \qquad (47)$$

Boundary condition (47) is applied at an unknown, moving position. To simplify the problem by having all the boundaries at known, stationary coordinates, we introduce

$$\xi(x, \hat{t}) \equiv \frac{S(\hat{t}) - x}{S(\hat{t})} \quad \text{and} \quad \eta(x, \hat{t}) \equiv \frac{x - S(\hat{t})}{1 - S(\hat{t})}$$

which transforms the interface position to $\xi = \eta = 0$. Next, we perturb the system about the known steady-state solution:

$$S(\hat{t}) = S_{\mathrm{ss}} + \delta \exp(\lambda \hat{t})$$

$$F(\hat{t}) = F_{\mathrm{ss}} + \left.\frac{\mathrm{d}F}{\mathrm{d}S}\right|_{S_{\mathrm{ss}}} \delta \exp(\lambda \hat{t})$$

$$\left.\begin{aligned}\theta_v(\xi,\hat{t}) &= \theta_{V_{ss}}(\xi) + f(\xi)\exp(\lambda\hat{t})\\ \theta_L(\eta,\hat{t}) &= \theta_{L_{ss}}(\eta) + g(\eta)\exp(\lambda\hat{t})\end{aligned}\right\} \quad (49)$$

where δ, $f(\xi)$ and $g(\eta)$ are small perturbations, and λ is an unknown complex constant. If $\lambda_{\text{real}} > 0$ the system is unstable; if $\lambda_{\text{real}} < 0$ it is stable. The above expressions are substituted into the transformed energy equations and boundary conditions. All terms of second order and higher in the perturbed quantities are neglected, yielding two linear, ordinary differential equations for $f(\xi)$ and $g(\eta)$. Solving and substituting into the energy balance boundary condition, we obtain the following transcendental equation for λ in terms of the independent parameters C, κ, H, Q, Pe_i, R, Y, E, V and the known steady-state solution S_{ss}.

$$\frac{Pe_iF_{ss}}{(1-S_{ss})(1-\mathrm{e}^{-\mu})}\left\{\left(\frac{W}{\lambda E}-1\right)\left[\frac{\sigma(1+\mathrm{e}^{-2\sigma})}{1-\mathrm{e}^{-2\sigma}}-\frac{\mu}{2}\right]-\frac{2\sigma W\,\mathrm{e}^{-((\mu/2)+\sigma)}}{E\lambda(1-\mathrm{e}^{-2\sigma})}\right\}+H(W-\varepsilon\lambda)$$
$$+\frac{\Gamma\,\mathrm{e}^{-\varDelta}\left\{\left(\frac{YW}{V\lambda}-D\right)\left[\left(\frac{\Delta}{2}+\tau\right)^2-\left(\frac{\Delta}{2}-\tau\right)^2\mathrm{e}^{-2\tau}\right]-2\tau\Delta\left(\frac{WY}{V\lambda}+1-D\right)\mathrm{e}^{[(\varDelta/2)-\tau]}\right\}}{\kappa S_{ss}\left[\left(\frac{\varDelta}{2}+\tau\right)-\left(\frac{\varDelta}{2}-\tau\right)\mathrm{e}^{-2\tau}\right]}=0 \quad (50)$$

where,

$$\sigma^2 \equiv (1-S_{ss})^2\left[\left(\frac{Pe_iF_{ss}}{2}\right)^2+E\lambda\right]; \qquad \mu \equiv (1-S_{ss})\,Pe_iF_{ss}^2$$

$$\tau^2 \equiv S_{ss}^2\left[\left(\frac{KPe_iF_{ss}}{2C}\right)^2+\frac{\kappa\lambda}{CY}\right]; \qquad W \equiv (R-1)\,Pe_iF_{ss}$$

$$\varDelta \equiv \frac{\kappa S_{ss}Pe_iF_{ss}}{C}; \qquad D \equiv 1+\frac{\varepsilon(Y-1)}{V}.$$

Equation (50) has been solved numerically by trial and error for several cases. It is found that when three steady-state solutions are possible, the middle position is unstable and the other two are stable. When $\lambda = 0$, two solutions merge into one which is metastable.

Transient problem—constant properties

The object of this section is to determine the rate at which the interface advances into the medium. An approximate solution will be obtained applying the heat-balance integral technique proposed by Goodman [3, 4]. According to this method, we assume a temperature profile in the liquid and vapor regions satisfying boundary conditions (5)–(7) and then integrate the governing energy equations. The advantage of this method is that it changes the energy equation from a partial to an ordinary differential equation; the disadvantage is that the energy equation is satisfied only on an average.

In choosing a temperature profile, consider the case where the rate of advance of the interface is slow enough so that changes in time can be neglected compared to changes in the space coordinate. For this "quasi-steady-state" case the temperature is dependent on time only implicity through the interface position. Hence, the assumed temperature profile will be taken in the form of the steady-state solution for $\theta_L(x, S)$ and $\theta_V(x, S)$.

Of particular interest is the solution of the transient interface position for the case where heat flux is prescribed on the boundary. For certain parameters this problem has two stable, steady-state interface positions. The initial conditions and time required to reach each steady-state position will be determined. The normalized, transient governing equations in the vapor and liquid regions are (45) and (46) respectively, subject to boundary conditions (35)–(37) and (47). Integrating equation (45) from $x = 0$ to $x = S(\hat{t})$ yields:

$$\left.\frac{\partial\theta_V}{\partial x}\right|_{S(\hat{t})} - \left.\frac{\partial\theta_V}{\partial x}\right|_0 + \frac{\kappa}{C}\left[Pe_iF + \varepsilon\left(1 - \frac{1}{Y}\right)\frac{\mathrm{d}S}{\mathrm{d}\hat{t}}\right] \times [\theta_V(S) - \theta_V(0)] = \frac{VK}{CY}\int_0^{S(\hat{t})}\frac{\partial\theta_V}{\partial\hat{t}}\,\mathrm{d}x. \quad (51)$$

Applying Leibnitz's rule to change the order of interaction and differentiation in the right hand side of equation (51), and substituting boundary conditions (35) and (37), we obtain:

$$\left.\frac{\partial\theta_V}{\partial x}\right|_{S(\hat{t})} + \Gamma + \frac{\kappa}{C}\left[Pe_iF + \varepsilon\left(1 - \frac{1}{Y}\right)\frac{\mathrm{d}S}{\mathrm{d}\hat{t}}\right] \times [1 - \theta_V(0)] = \frac{V\kappa}{CY}\left(\frac{\mathrm{d}\psi_V}{\mathrm{d}\hat{t}} - \frac{\mathrm{d}S}{\mathrm{d}\hat{t}}\right) \quad (52)$$

where,

$$\psi_V(\hat{t}) \equiv \int_0^{S(\hat{t})} \theta_V(x_1\hat{t})\,\mathrm{d}x. \quad (53)$$

where,

$$\psi_L(\hat{t}) \equiv \int_{S(\hat{t})}^1 \theta_L(x, \hat{t})\,\mathrm{d}x. \quad (55)$$

Combining equations (52), (54) and (47) we obtain the energy balance boundary condition as:

$$Pe_iF\left[H + 1 - \frac{1 - \theta_V(0)}{C}\right] - \left.\frac{\partial\theta_L}{\partial x}\right|_{x=1} + E\frac{\mathrm{d}\psi_L}{\mathrm{d}\hat{t}} + \frac{V}{CY}\frac{\mathrm{d}\psi_V}{\mathrm{d}\hat{t}} + \left[E - \frac{V}{CY} + \varepsilon H + \frac{\varepsilon}{C}\left(1 - \frac{1}{Y}\right) \times (1 - \theta_V(0))\right]\frac{\mathrm{d}S}{\mathrm{d}\hat{t}} = \frac{\Gamma}{\kappa}. \quad (56)$$

Substituting the assumed temperature profiles of the liquid and vapor region, equations (32) and (39) respectively, into equations (53), (55) and (56) yields the following first order, ordinary differential equation for the interface position:

$$\Bigg[\!\!\Bigg[E - \frac{EPe_iRF^2(1 - S)}{\exp[Pe_iF(1 - S)] - 1} + \{1 - \exp[-Pe_iF(1 - S)]\}\Bigg\{\frac{E(R - 1)}{Pe_i} + \varepsilon H + \frac{V\Gamma S}{CY}\exp(-\kappa Pe_iFS/C)\left[1 + (R - S)\left(FS + \frac{2C}{\kappa Pe_i}\right)\right] + \frac{\Gamma}{\kappa Pe_iE}[1 - \exp(-\kappa Pe_iFS/C)]\left[\varepsilon\left(1 - \frac{1}{Y}\right) + \frac{2V(R - 1)C}{Y\kappa Pe_i}\right]\Bigg\}\Bigg]\!\!\Bigg]\frac{\mathrm{d}S}{\mathrm{d}\hat{t}} = \{1 - \exp[-Pe_iF(1 - S)]\}\left\{\frac{\Gamma\exp(-\kappa Pe_iFS/C)}{\kappa} - Pe_iFH\right\} - Pe_iF. \quad (57)$$

Integrating equation (46) from $x = S(\hat{t})$ to $x = 1$, applying Leibnitz's rule and boundary conditions (36) and (37) yields:

$$\left.\frac{\partial\theta_L}{\partial x}\right|_{x=1} - \left.\frac{\partial\theta_L}{\partial x}\right|_{S(\hat{t})} - Pe_iF = E\left(\frac{\mathrm{d}\psi_L}{\mathrm{d}\hat{t}} + \frac{\mathrm{d}S}{\mathrm{d}\hat{t}}\right) \quad (54)$$

A fourth order Runge–Kutta method [27] has been applied in the numerical solution of the above equation.

Typical results for the transient interface position are presented in Fig. 7. For the cases where three steady-state interface positions are found, let us denote the position near $x = 0$ and

S_{1ss}, the middle position as S_{2ss}, and the one near the reservoir as S_{3ss}. If the initial condition is such that $0 \leqslant S\ (t = 0) < S_{2ss}$, the interface approaches S_{1ss}. If $S_{2ss} < S\ (t = 0) \leqslant 1$, the interface approaches S_{3ss}. When there is only one steady-state position near the reservoir end, the smaller the initial Reynolds number or Péclét number the faster the interface penetrates into the medium. (See curves with $q_{in}/q_{min} = 10$ in Fig. 7.) Since the assumed temperature profile was taken in the form of the steady-state solution, it is expected that the smaller $|dS/dt|$ (higher Pe_i), the better the accuracy of these results.

SUMMARY AND CONCLUSIONS

An analysis of two phase flow in porous media has been presented. A phase change interface separating a vapor region from the liquid region has been induced by heating one end of the porous medium above the vaporizing temperature of the liquid. In general, it has been found that for a large initial Péclét number (convection dominated flow) the interface does not penetrate into the system. This is because most of the energy entering is convected out with the vapor flux. On the other hand, for low Pe_i, the interface is able to advance into the medium, and we obtain a linear temperature profile in the vapor and liquid regions.

Several other main points arise from the analysis of this problem. They are:

1. Limits on the solution can be obtained by applying upper and lower bounds on the effective thermal conductivity of the porous medium. Bounds on the solution are smaller for either convection dominated ($Pe_i \gg 1$) or conduction dominated ($Pe_i \ll 1$) flow than for flow with both convection and conduction. (See Fig. 5.) Therefore, the accurate prediction of the effective conductivity is most important for initial Péclét number around order one.
2. If a constant temperature is prescribed on the boundary there is only one steady-state interface position; but if the heat flux is prescribed, there may be three steady-state positions, two of which are stable.
3. If more than one steady-state solution is possible, the initial conditions determine which one is reached. For conduction dominated flow where the interface can penetrate into the medium, the smaller the initial Péclét number the faster the system approaches a steady state.
4. The results of the constant and variable property cases (see Appendix II) give the same effects. Therefore, to determine the influence of convection, conduction, and phase change a study of the constant property case is adequate.
5. An evaluation of the validity of this theory requires a comparison with experimental results. To the authors' knowledge, no such experiments have been reported in the literature. An asymptotic solution for large time is presently being investigated to determine the accuracy of the heat-balance integral technique for the transient problem.

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APPENDIX I

Determination of Minimum Heat Input

First, we shall determine the heat flux per unit area necessary to raise the temperature of the system at $x = 0$ to the saturation temperature. The governing energy equation (24) is subject to the boundary conditions:

$$\text{at} \quad x = 0: \qquad q^* = -k_{L_{\mathrm{eff}}} \frac{\mathrm{d}T}{\mathrm{d}x}$$

$$\text{at} \quad x = L: \qquad T = T_R$$

where q^* is the heat flux necessary to raise the temperature at $x = 0$ to T^*. Integrating equation (24) yields the temperature distribution,

$$T(x) = T_R - \frac{q^*(\mathrm{e}^{-Pe_i x} - \mathrm{r}^{-Pe_i})}{\dot{m}_i C_{P_L}}.$$

Evaluating at $x = 0$, and rearranging terms we obtain q^* as:

$$q^* = -\frac{\dot{m}_i[h_L(T^*) - h_L(T_R)]}{(1 - \mathrm{e}^{-Pe_i})}$$

where $h_L(T^*) - h_L(T_R) = C_{P_L}(T^* - T_R)$. An additional heat flux equal to $\dot{m}_i h_{fg}(T^*)$ is required to vaporize the liquid. Adding this heat flux to q^* we obtain the minimum heat input.

$$q_{\min} = \frac{Pe_i k_{L_{\mathrm{eff}}}}{LC_{P_L}} \left[h_{fg}(T^*) + \frac{h_L(T^*) - h_L(T_R)}{1 - \mathrm{e}^{-Pe_i}} \right].$$

Taking limits on Pe_i.

as $Pe_i \to 0$:

$$q_{\min} = \frac{k_{L_{\mathrm{eff}}}}{L} \left[T^* - T_R + \frac{Pe_i h_{fg}(T^*)}{C_{P_L}} \right] + 0\,(Pe_i^2)$$

as $Pe_i \to \infty$

$$q_{\min} = \frac{Pe_i k_{L_{\mathrm{eff}}}}{L} \left[\frac{h_{fg}(T^*)}{C_{P_L}} + T^* - T_R \right] + 0\left(\frac{1}{\mathrm{e}^{Pe_i}}\right)$$

APPENDIX II

Steady-state, Variable Properties, Constant Temperature Boundary Condition

The purpose of this section is to compare the results of the constant property case to the case with temperature dependent kinematic viscosity, since ν_L is the most temperature sensitive property.† The procedure for a constant temperature boundary condition is identical to that of the case with a constant surface heat flux [28].

In obtaining P^* as a function of S, we take basically the same approach as in the constant property case, and substituting for P^* in equation (21) yields:

$$\dot{m}_L = -\frac{K(P_R - P_0)}{v_L(L - S) + \int_0^s v_V(T)\,\mathrm{d}x}. \qquad \text{(A.1)}$$

For convenience, the expression relating the kinematic viscosity, v_V and temperature is taken in a polynomial form,‡

$$v_V(T) = A_0 + A_1(T - T^*) + A_2(T - T^*)^2 + A_3(T - T^*)^3 \qquad \text{(A.2)}$$

where the A_i's are constants.

The governing equations for the temperature are the same as those for the constant property case except the normalized flow rate, F, is not a known function of the interface position. Dividing $\dot{m}_L$ in equation (A.1) by $\dot{m}_i$, and combining with the polynomial expression for the kinematic viscosity, we obtain the normalized flow rate,

$$F = \frac{1}{1 - S + \int_0^s [V_0 + \theta_0 V_1(1 - \theta_V) + \theta_0^2 V_2(1 - \theta_V)^2 + \theta_0^3 V_3(1 - \theta_V)^3]\,\mathrm{d}x} \qquad \text{(A.3)}$$

where,

$$V_0 \equiv \frac{A_0}{v_L},\ V_1 \equiv \frac{A_1(T^* - T_R)}{v_L},\ V_2 \equiv \frac{A_2(T^* - T_R)^2}{v_L},\ V_3 \equiv \frac{A_3(T^* - T_R)^3}{v_L}.$$

With $\theta_V(x)$ known from equation (31), the integral in the above expression may be integrated to yield:

$$Z = \frac{\kappa Pe_i}{C} \Big/ \left\{1 + S(V_0 - 1) + \theta_0 V_1\left(\frac{1}{Z} - \frac{S}{e^{ZS} - 1}\right) + \frac{\theta_0^2 V_2(e^{2ZS} - 4e^{ZS} + 2ZS + 3)}{2Z(e^{ZS} - 1)^2} + \frac{\theta_0 V_3(2e^{3ZS} - qe^{2ZS} - 18e^{ZS} - 11 - 6ZS)}{6Z(e^{ZS} - 1)^3}\right\} \qquad \text{(A.4)}$$

where $Z \equiv \kappa Pe_i F/C$. Only for $V_1 = V_2 = V_3 = 0$, which is equivalent to $v_V =$ constant, one may obtain F as an explicit function of S. We obtain another equation in F and S by substituting the normalized temperatures from equations (31) and (32) into the interface energy equation (30).

$$\frac{\theta_0}{C} = [\exp(\kappa Pe_i FS/C) - 1] \left\{H + \frac{1}{1 - \exp[-Pe_i F(1 - S)]}\right\}. \qquad \text{(A.5)}$$

Equations (A.4) and (A.5) are two, coupled transcendental equations for F and S. A solution is obtained by applying a modified Newton–Raphson iteration technique [29]. The substitution of F and S into equations (31) and (32) yields the complete solution for the temperature distribution.

Figure 8 presents a comparison of a typical temperature profile of the constant property and variable property cases. In both the liquid and vapor regions the Péclét number is small; hence the temperature distribution is linear. Figures 9 and 10 compare the two cases for the steady-state interface position and the normalized flow rate respectively. These results show that the constant property case predicts the same effects as the variable property case even if v_V/v_L varies by a factor of 16 for constant properties. This is readily seen in Fig. 8. Therefore, since we are mainly interested in studying the influence of convection, conduction, and phase change on the interface position, the constant property case yields adequate results for this purpose.

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† For example, the kinematic viscosity of steam at atmospheric pressure varies by about a factor of four between 100°C and 500°C, whereas the thermal conductivity varies by less than half as much, and the specific heat varies only 2 per cent [26]. Other gases exhibit similar behavior.

‡ Comparing values of the kinematic viscosity for saturated steam tabulated in [26] shows that a second order polynomial approximation for $v(T)$ has an accuracy of ±6 per cent for the temperature range 100–800°C whereas the third order polynomial, equation (A.2), is accurate to ±2 per cent in the same temperature range.

TRANSFERT THERMIQUE DANS UN MILIEU POREUX AVEC CHANGEMENT DE PHASE

Résumé—Un liquide provenant d'un réservoir à température constante traverse un milieu poreux. A l'extrémité opposée la température du système est portée au-dessus du point d'ébullition du liquide en imposant soit une température constante soit un flux thermique constant. Ceci provoque la propagation d'un front d'évaporation dans le matériau avec séparation en une région de vapeur et une région de liquide. L'objet de cet article et d'analyser le système en recherchant l'importance relative de la convection par rapport à la conduction et en dégageant les principaux paramètres influençant la distribution de température et la position de l'interface. Des solutions exactes sont obtenues pour le problème permanent où (1) les propriétés sont constantes et (2) la densité et la viscosité de la vapeur est variable. On obtient aussi une solution approchée du problème transitoire avec propriétés constantes en appliquant la technique intégrale du bilan thermique.

WÄRMETRANSPORT IN PORÖSEN MEDIEN MIT PHASENÜBERGANG

Zusammenfassung—Flüssigkeit aus einem Reservoir mit konstanter Temperatur gelangt in ein poröses Material. Auf der freien Seite wird die Temperatur des Systems über der Siedetemperatur der Flüssigkeit gehalten durch Aufprägen einer konstanten Temperatur oder eines konstanten Wärmeflusses. Dies ruft eine Verdampfungsfront hervor, die in das Material vordringt und so einen Dampfbereich von dem Flüssigkeitsbereich abtrennt. In dem Artikel soll dieses System untersucht werden: die Bedeutung der Konvektion in Verhältnis zur Wärmeleitung, die Hauptparameter, die die Temperaturverteilung beeinflussen, und die Lage der Phasengrenze.

Exakte Lösungen wurden für das stationäre Problem gefunden, wobei (1) die Eigenschaften konstant sind und (2) Dichte und Viskosität des Dampfes veränderlich sind. Auch wurde eine Näherungslösung für das instationäre Problem gefunden, bei konstanten Eigenschaften, durch Anwendung einer integralen Energiebilanz.

ПЕРЕНОС ТЕПЛА В ПОРИСТЫХ СРЕДАХ ПРИ НАЛИЧИИ ФАЗОВОГО ИЗМЕНЕНИЯ

Аннотация—Жидкость из резервуара с постоянной температурой поступает в пористую среду. На противоположном конце температура системы выше температуры кипения жидкости, что достигается поддержанием постоянной температуры или постоянного теплового потока. Это вызывает распростанение фронта испарения внутри материала, отделяя область пара от области жидкости. В данной работе ставилось целью проанализировать эту систему, исследуя относительную влажность конвекции по сравнению с кондукцией и основные параметры, влияющие на распределение температуры и положение границы раздела фаз. Получены точные решения для стационарной задачи, где (1) свойства являются постоянными, а (2) плотность и вязкость пара переменными. Получено приближенное решение нестационарной задачи с постоянными свойствами интегрирования баланса тепла.