Transp Porous Med (2012) 92:1–14
DOI 10.1007/s11242-011-9883-x

Unsteady mixed convection boundary-layer flow with suction and temperature slip effects near the stagnation point on a vertical permeable surface embedded in a porous medium

Azizah Mohd Rohni · Syakila Ahmad · Ioan Pop · John H. Merkin

Received: 2 August 2011 / Accepted: 20 September 2011 / Published online: 14 October 2011

Abstract The unsteady mixed convection boundary-layer flow near the two-dimensional stagnation point on a vertical permeable surface embedded in a fluid-saturated porous medium with suction and a temperature slip effect is studied numerically. Similarity equations are obtained through the application of a similarity transformation technique. The shooting method is used to solve these similarity equations for different values of the mixed convection, wall mass suction, the unsteadiness and the slip parameters. Results show that multiple solutions exist for certain ranges of these parameters. Some limiting forms are then discussed, namely strong suction, the free convection limit, the situation when there is a large temperature slip and when the time dependence dominates.

Keywords Porous medium · Unsteady · Mixed convection · Suction · Numerical results

1 Introduction

The transport of heat through a porous medium is an active field of research as it plays a crucial role in many diverse applications. Considerable work has been reported on the flow, heat and mass transfer in Darcian porous media and has been the subject of various studies because of the increasing need for a better understanding of the associated transport processes.

A. M. Rohni
UUM College of Arts & Sciences, Physical Science Division, Building of Quantitative Sciences, Universiti Utara Malaysia, 06010 Sintok, Kedah, Malaysia

S. Ahmad
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia

I. Pop (✉)
Faculty of Mathematics, University of Cluj, P.O. Box CP 253, 400082 Cluj, Romania
e-mail: popm.ioan@yahoo.co.uk

J. H. Merkin
Department of Applied Mathematics, University of Leeds, Leeds, LS2 9JT, UK

This interest stems from the numerous practical applications which can be modelled or can be approximated as transport through porous media such as packed beds, high performance insulation for buildings, chemical catalytic reactors, grain storage, migration of moisture through the air contained in fibrous insulations, heat exchange between soil and atmosphere, heat storage beds and beds of fossil fuels such as oil shale and coal, salt leaching in soils, solar power collectors, electrochemical processes, insulation of nuclear reactors, regenerative heat exchangers and geothermal energy systems and many other areas. Literature concerning flow in porous media is abundant. Representative studies in this area may be found in the recent books by Nield and Bejan (2006), Pop and Ingham (2001), Ingham and Pop (2005), Vafai (2005, 2010) and Vadasz (2008).

Steady mixed convection boundary-layer flow past inclined or vertical flat surfaces placed in a porous medium has been investigated by Cheng (1977), Merkin (1980, 1985) and Aly et al. (2003). Cheng (1977) obtained similarity solutions for the situation where the free stream velocity and the surface temperature distribution vary according to the same power function of the distance along the surface. Numerical and asymptotic solutions have been given for both aiding flows, where the flow is directed vertically upwards, and opposing flows, where the flow is directed vertically downwards, i.e. where the buoyancy and inertia forces act in the same direction (aiding) or in the opposite direction (opposing). For opposing flows, the numerical solution breaks down at a finite (negative) value of the mixed convection parameter. Hence, the boundary layer may separate from the surface, giving rise to rather unusual heat transfer characteristics. It appears that the separation in mixed convection flow in porous media was first discussed by Merkin (1980, 1985), who examined the effect of opposing buoyancy forces on the boundary-layer flow on a semi-infinite vertical flat surface at a constant temperature in a uniform free stream. Aly et al. (2003) assumed that both the temperature of the surface and the fluid velocity at the edge of the boundary layer vary as x^m, where x is the distance from the leading edge of the plate and m is a preassigned constant so that a similarity solution can be obtained. Finally, we mention that Harris et al. (2009) have studied the steady mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium with slip using the Brinkman model.

In general, the unsteady Navier–Stokes equations can be solved only by using numerical integrations of the partial differential equations. However, if we restrict the motion to a specified family of time dependence, exact similarity solutions can be obtained as was shown originally by Yang (1958) and Birkhoff (1960). The form for the outer flow given by the relation $u_e(t, x) = ax/(1 - \gamma t)$ used here and referred to as 'hyperbolic time variation' by Yang (1958), has been also used by many researchers (for example, Wang 1990; Andersson et al. 2000; Ali and Magyari 2007; Tie-Gang et al. 2009). Nazar et al. (2004) have studied the unsteady mixed convection stagnation point flow on a vertical surface in a fluid-saturated porous medium. The main concern of their work centred on the time-dependent behaviour in the neighbourhood of the stagnation point on the vertical flat surface using the Keller-box method. The variation of the skin friction or shear stress parameter as a function of the mixed convection parameter λ reveals dual solutions in the parameter range $\lambda_c < \lambda < -1$, with λ_c being the critical value of λ, where the boundary-layer solution breaks down. In a more recent paper, Merrill et al. (2006) have further studied Nazar's et al. (2004) boundary-value problem by a detailed mathematical and numerical analysis. They have proved the existence of a solution to the governing boundary-value problem for all $\lambda > -1$ and presented numerical evidence that a second solution exists for $\lambda > -1$, thus giving dual solutions for all $\lambda > \lambda_c$. It has been also proved that, if $\lambda < -2.9136$, no solution to the boundary-value problem exists. Finally, a stability analysis has been performed by Merrill et al. (2006) to show that solutions on the upper branch are linearly stable, while those on the lower branch are unstable. We also mention to this end the papers by Harris et al. (1996, 2002) on the transient boundary-layer flow past a vertical semi-infinite flat plate embedded in a porous medium.

Our interest in the present paper is to study the unsteady mixed convection boundary-layer flow near the two-dimensional stagnation point on a vertical permeable surface embedded in a fluid-saturated porous medium with suction and a temperature slip effect which we reduce to similarity form. The shooting method is used to solve the resulting similarity or ordinary differential equations for different values of the mixed convection, wall mass suction, the unsteadiness and the slip parameters.These results show that multiple solutions exist for a certain range of these parameters. We then go on to discuss some limiting forms, namely strong suction indicating the possibility of reversed flow, the free convection limit with nature of this limit being dependent on whether there is a temperature slip or not, the situation when there is a large temperature slip and when the time dependence dominates. In these latter two cases, it is the outer flow which is the dominant feature. We start by deriving our model.

2 Equations

We consider the unsteady mixed convection boundary-layer flow near the stagnation point on a heated permeable vertical surface embedded in a saturated porous medium, as shown in Fig. 1, where the x and y axes are measured along the surface and normal to it, respectively. The surface coincides with the plane $y = 0$ and the flow takes place in the region $y \geq 0$. The velocity of the flow far from the surface is $u_{\mathrm{e}}(\bar{t}, x)$, where $\bar{t}$ is time. It is assumed that the temperature of the surface is $T_{\mathrm{w}}(\bar{t}, x)$ and that the ambient fluid is at the constant temperature T_∞, where $T_{\mathrm{w}}(\bar{t}, x) > T_\infty$ for an assisting flow, while $T_{\mathrm{w}}(\bar{t}, x) < T_\infty$ for an opposing flow, respectively. The forms of $u_{\mathrm{e}}(\bar{t}, x)$ and $T_{\mathrm{w}}(\bar{t}, x)$ will be defined below. Under the assumptions of Darcy's law and the Boussinesq approximation, the basic unsteady boundary-layer equations of the problem under consideration are seen in Nield and Bejan (2006)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u = u_{\mathrm{e}} + \frac{gK\beta}{\nu}(T - T_\infty) \tag{2}$$

$$\sigma \frac{\partial T}{\partial \bar{t}} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{\mathrm{m}} \frac{\partial^2 T}{\partial y^2} \tag{3}$$

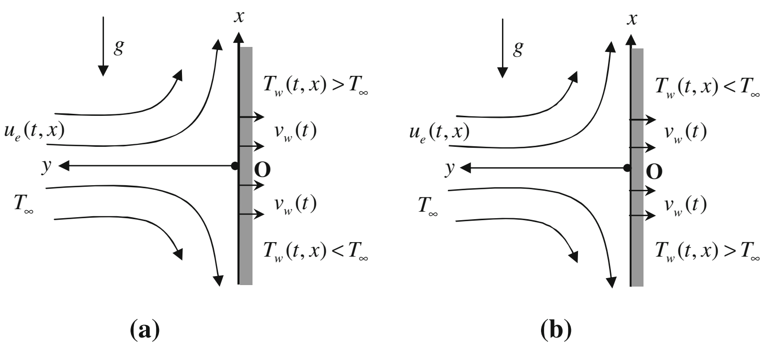

Fig. 1 A schematic representation of our model for **a** assisting flow and **b** opposing flow

where u and v are the velocity components in the x and y directions, T is the fluid temperature, K is the permeability of the porous medium, g is the acceleration due to gravity, ν is the kinematic viscosity of the convecting fluid, α_{m} is the effective thermal diffusivity and σ is the heat capacity ratio of the fluid-filled porous medium to that of the fluid. In order to eliminate σ from Eq. 3, we take $t = \bar{t}/\sigma$. We assume that Eqs. 1–3 are subject to the initial and boundary conditions that

$$\begin{aligned} u = v = 0, \quad & T = T_\infty \quad \text{at } t = 0, \quad (\text{for all } x, y) \\ v = v_{\mathrm{w}}(t), \quad & T = T_{\mathrm{w}} + N_1 \frac{\partial T}{\partial y} \quad \text{at } y = 0 \quad (t > 0) \\ u \to u_{\mathrm{e}}, \quad & T \to T_\infty \quad \text{as } y \to \infty \quad (t > 0) \end{aligned} \tag{4}$$

where $v_{\mathrm{w}}(t) < 0$ is the suction velocity. We assume that $u_{\mathrm{e}}(t,x)$, $T_{\mathrm{w}}(t,x)$ and $N_1(t)$ have the form

$$u_{\mathrm{e}}(t,x) = \frac{ax}{1-\gamma t}, \quad T_{\mathrm{w}}(t,x) = T_\infty + T_0 \frac{x}{1-\gamma t}, \quad N_1 = N_0(1-\gamma t)^{1/2} \tag{5}$$

so that the system (1–4) can be reduced to similarity form. Here N_0 is the initial value of the thermal slip factor and T_0 is the characteristic temperature, both assumed constant, for a heated surface (assisting flow) $T_0 > 0$ and $T_0 < 0$ for a cooled surface (opposing flow), respectively. In addition, $a > 0$ and γ are constants. The form of the temperature slip factor $N_1(t)$ has been recently proposed by Mukhopadhyay and Andersson (2009).

We look for a similarity solution of Eqs. 1–3 in the following form:

$$\psi = \sqrt{\frac{a\alpha_{\mathrm{m}}}{1-\gamma t}}\, x\, f(\eta), \quad \theta = \frac{T - T_\infty}{T_{\mathrm{w}} - T_\infty}, \quad \eta = y\sqrt{\frac{a}{\alpha_{\mathrm{m}}(1-\gamma\, t)}} \tag{6}$$

where ψ is the stream function (defined in the usual way as $u = \psi_y$ and $v = -\psi_x$). Thus, we have

$$u = \frac{ax}{1-\gamma t} f'(\eta), \quad v = -\sqrt{\frac{a\alpha_{\mathrm{m}}}{1-\gamma t}}\, f(\eta) \tag{7}$$

where primes denote differentiation with respect to η. We note here that, in general, primes denote differentiation with respect to function argument. Hence from (7), we take

$$v_{\mathrm{w}}(t) = -s\sqrt{\frac{a\alpha_{\mathrm{m}}}{1-\gamma\, t}} \tag{8}$$

where $s > 0$ is the constant suction parameter, again to ensure a similarity form.

Substituting (6) into Eqs. 2 and 3, we obtain

$$f' = 1 + \lambda\theta \tag{9}$$

$$\theta'' + f\theta' - f'\theta - A\left(\theta + \frac{\eta}{2}\theta'\right) = 0 \tag{10}$$

subject to the boundary conditions that

$$f(0) = s, \quad \theta(0) = 1 + \Lambda\,\theta'(0), \quad \theta \to 0, \quad f' \to 1 \quad \text{as} \quad \eta \to \infty \tag{11}$$

The dimensionless parameters A, λ and Λ are the flow unsteadiness, mixed convection and thermal slip parameters, respectively, which are defined as

$$A = \frac{\gamma}{a}, \quad \lambda = \frac{Ra_x}{Pe_x} = \frac{g\beta K T_0}{\nu a}, \quad \Lambda = N_0\left(\frac{a}{\alpha_{\mathrm{m}}}\right)^{1/2} \tag{12}$$

where $Ra_x = \dfrac{gK\beta(T_w - T_\infty)\,x}{\alpha_m \nu}$ is the local Rayleigh number for a porous medium and $Pe_x = \dfrac{u_e\,x}{\alpha_m}$ is the local Péclet number. For the present situation, we assume that $A \le 0$, i.e. $\gamma \le 0$ (see Fang et al. 2009) to ensure that a singularity does not arise in the forms for $\psi,\ \theta,\ \eta$ in (6) at a finite time. It is worth mentioning that $\lambda > 0$ corresponds to assisting flow (heated surface), $\lambda < 0$ to opposing flow (cooled surface) and $\lambda = 0$ to forced convection flow ($T_0 = 0$).

Equations 9 and 10 can be combined to give the single equation

$$f''' + ff'' + f'(1 - f') - A\left(f' - 1 + \frac{\eta}{2} f''\right) = 0 \tag{13}$$

with boundary conditions (11) becoming

$$f(0) = s, \quad f'(0) = 1 + \lambda + \Lambda\, f''(0), \quad f' \to 1 \quad \text{as} \quad \eta \to \infty \tag{14}$$

We note that, when $A = \Lambda = s = 0$, Eq. 13 with the corresponding boundary conditions (14) reduces to the steady-state equation 25 in the paper by Harris et al. (2009) and also for the same case it reduces to Eq. 21 in the paper by Nazar et al. (2004).

The quantities of physical interest in this problem are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_e^2}, \quad Nu_x = \frac{x\, q_w}{k(T_w - T_\infty)} \tag{15}$$

where $\tau_w = \mu \left(\dfrac{\partial u}{\partial y}\right)_{y=0}$ is the skin friction or shear stress at the surface and $q_w = -k\left(\dfrac{\partial T}{\partial y}\right)_{y=0}$ is the heat flux from the surface, where μ is the dynamic viscosity and k is the thermal conductivity of the fluid. Using (6), (7) and (14), we find that

$$\tau_w = \mu \left(\frac{a}{1 - \gamma t}\right)^{3/2} \frac{x}{\alpha_m^{1/2}}\, f''(0), \quad q_w = -\frac{kT_0 x}{(1 - \gamma t)^{3/2}} \left(\frac{a}{\alpha_m}\right)^{1/2} \theta'(0) \tag{16}$$

3 Results

Numerical solutions to the governing ordinary differential equation (13) subject to the boundary conditions (14) were obtained using a standard shooting method. Results are reported for values of the mixed convection parameter λ, both positive and negative, the suction parameter $s \ge 0$, the unsteadiness parameter $A \le 0$ and the temperature slip parameter Λ.

In Fig. 2, we plot $f''(0)$ and $-\theta'(0)$ against λ for $s = 0$, $\Lambda = 0$ and $A = 0$, corresponding to no temperature slip and steady flow. In this case, we find a critical value $\lambda_c \simeq -1.4174$ of λ in agreement with the results of Nazar et al. (2004) and Merrill et al. (2006), thus giving confidence in our numerical approach. The forced convection, $\lambda = 0$, solution is simply $f = \eta + s$, giving $f''(0) = 0$ as can be seen in Fig. 2a. For $\lambda_c \le \lambda < 0$, $f''(0) > 0$ whereas for $\lambda > 0$, $f''(0) < 0$ decreasing as λ is increased on the upper solution branch. $\theta'(0) < 0$ on the upper solution branch, though $\theta'(0) > 0$ on the lower solution branch indicating a temperature overshoot.

For $\lambda = 0$, Eq. 10 gives

$$\theta'' + \left(\frac{2 + |A|}{2}\eta + s\right)\theta' - (1 - |A|)\theta = 0 \tag{17}$$

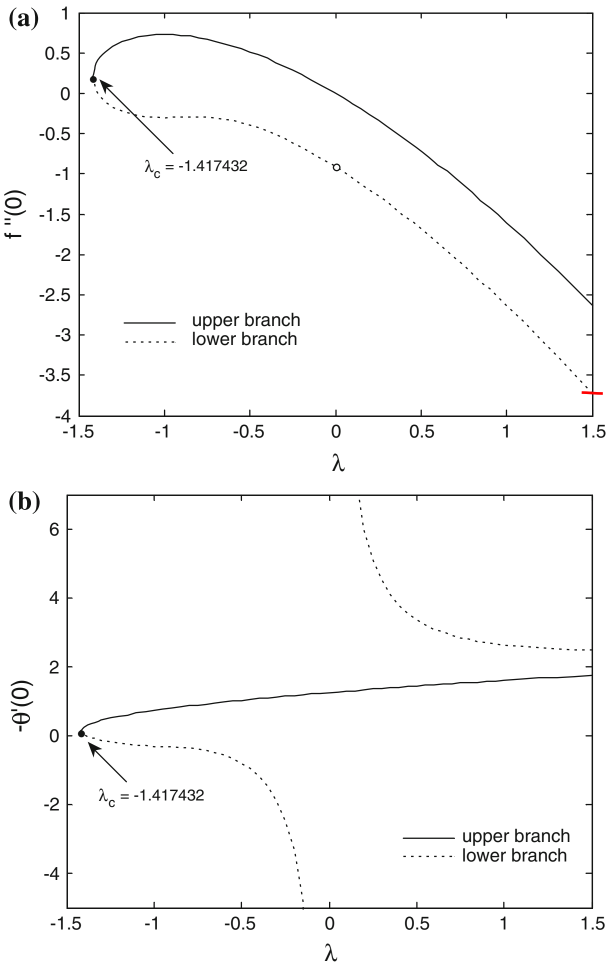

Fig. 2 Plots of **a** $f''(0)$ and **b** $-\theta'(0)$ against the mixed convection parameter λ for $s = 0$, $\Lambda = 0$ (no temperature slip), $A = 0$ (steady flow) obtained from the numerical solution of (13, 14). The disruption in $f''(0)$ in the numerical solutions as they pass through $\lambda = 0$ is indicated by $\circ$

still subject to (11). Equation 17 can be solved in terms of confluent hypergeometric and gamma functions (Slater 1960) as, on restricting attention to the case $s = 0$ and noting that $A \leq 0$,

$$\theta = A_0 e^{-(2+|A|)\eta^2/4}\, U\left(\frac{4-|A|}{2(2+|A|)};\ \frac{1}{2};\ \frac{2+|A|}{4}\,\eta^2\right) \quad (|A| \neq 4) \tag{18}$$

where

$$A_0^{-1} = \sqrt{\pi}\left(\frac{1}{\Gamma\left(\frac{3}{2+|A|}\right)} + \frac{\Lambda(2+|A|)^{1/2}}{\Gamma\left(\frac{4-|A|}{2(2+|A|)}\right)}\right)$$

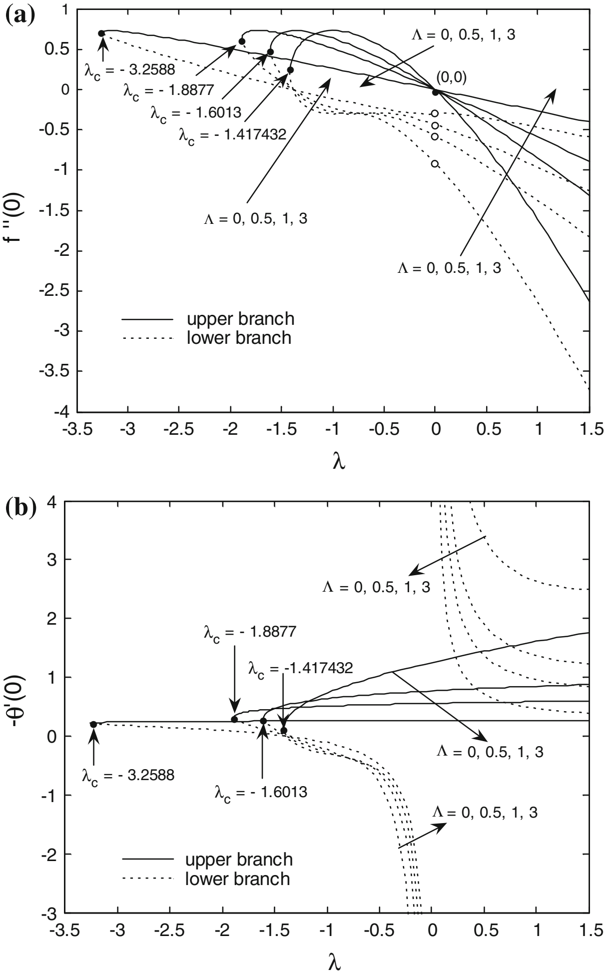

Fig. 3 Plots of **a** $f''(0)$ and **b** $-\theta'(0)$ against the mixed convection parameter λ for $s = 0$, $A = 0$ (steady flow) and $\Lambda = 0,\ 0.5,\ 1.0,\ 3.0$ obtained from the numerical solution of (13, 14). The disruption in $f''(0)$ in the numerical solutions as they pass through $\lambda = 0$ is indicated by $\circ$

giving

$$\theta'(0) = -\frac{A_0\sqrt{\pi}(2+|A|)^{1/2}}{\Gamma\left(\frac{4-|A|}{2(2+|A|)}\right)} \quad (|A| \neq 4) \tag{19}$$

When $|A| = 4$, Eq. 17 and boundary condition (11) give, still with $s = 0$,

$$\theta = \exp(-3\eta^2/2), \quad \theta'(0) = 0$$

For $A = \Lambda = 0$, expression (19) gives $\theta'(0) = -\sqrt{\pi/2}$ in agreement with the value shown in Fig. 2b.

In Fig. 3, we assess the effect of increasing the temperature slip parameter Λ, again for $s = 0$ and steady flow ($A = 0$). As in Fig. 2, in all cases $f''(0) > 0$ when $\lambda_c \leq \lambda < 0$ and

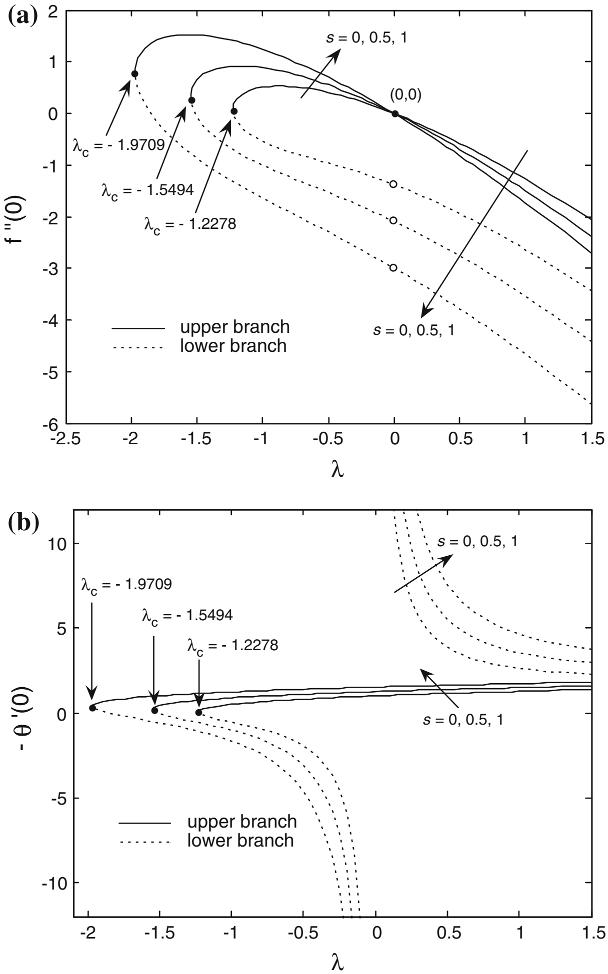

Fig. 4 Plots of **a** $f''(0)$ and **b** $-\theta'(0)$ against the mixed convection parameter λ for $\Lambda = 0.1$, $A = -0.5$ and $s = 0.0,\ 0.5,\ 1.0$ obtained from the numerical solution of (13, 14). The disruption in $f''(0)$ in the numerical solutions as they pass through $\lambda = 0$ is indicated by $\circ$

$f''(0) < 0$ when $\lambda > 0$ decreasing as λ is increased. Also $\theta'(0) < 0$ on the upper solution branch for all λ also decreasing, though slightly less rapidly, as λ is increased. However, the main effect of the slip parameter Λ is to decrease the value of λ_c, thus increasing the range of λ where there are dual solutions in the opposing flow case.

The effect of the suction parameter s is seen in Fig. 4 where we give $f''(0)$ and $-\theta'(0)$ plotted against λ for an unsteady flow with $A = -0.5$ and with a temperature slip, having $\Lambda = 0.1$. This figure clearly shows that the effect of having fluid withdrawal through the surface is to decrease the critical value λ_c, again giving a greater range of negative λ for solutions in the opposing flow case. As seen previously on the upper solution branch, $f''(0) > 0$ for $\lambda < 0$ and $f''(0) < 0$ for $\lambda > 0$ decreasing as λ is increased. Also, $\theta'(0) < 0$ on the upper solution branch.

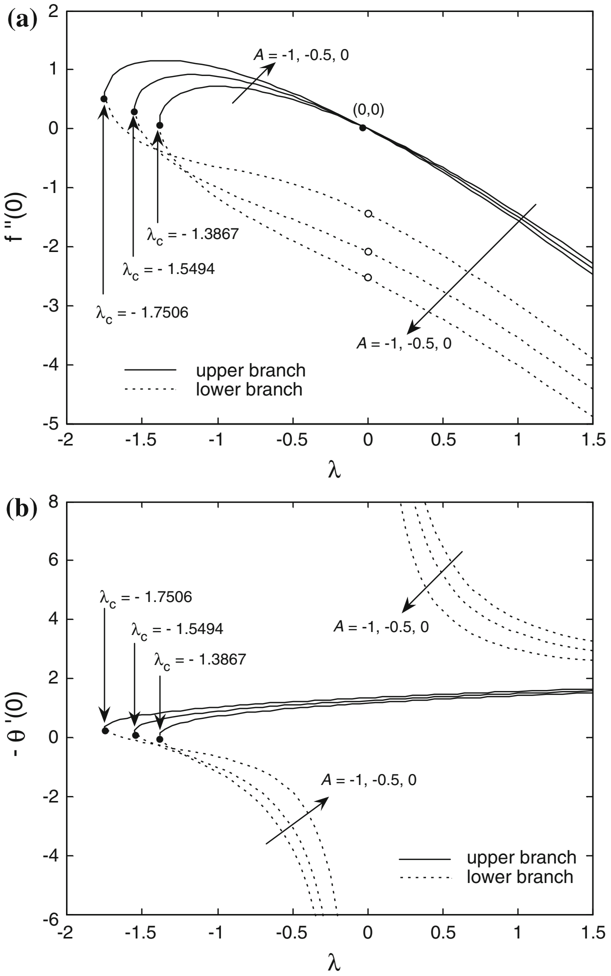

Fig. 5 Plots of **a** $f''(0)$ and **b** $-\theta'(0)$ against the mixed convection parameter λ for $s = 0.5$, and $\Lambda = 0.1$ and $A = 0.0, -0.5, -1.0$ obtained from the numerical solution of (13, 14). The disruption in $f''(0)$ in the numerical solutions as they pass through $\lambda = 0$ is indicated by $\circ$

The influence of the unsteadiness parameter A is seen in Fig. 5 where we plot $f''(0)$ and $-\theta'(0)$ against λ for $s = 0.5$ and $\Lambda = 0.1$. Again we see that, on the upper solution branch, $f''(0)$ changes sign at $\lambda = 0$ and that $\theta'(0) < 0$. However, the main effect of increasing the unsteadiness parameter is to decrease the critical value λ_c, thus reducing the range of solutions in the opposing flow case.

Finally, we comment on the nature of the solutions on the lower branch as they pass through $\lambda = 0$. Clearly, when $\lambda = 0$ there is only one solution $f = \eta + s$ giving $f''(0) = 0$, the value on the upper branch. Thus, the numerical solutions on the lower branch are disrupted at $\lambda = 0$, though the values of $f''(0)$ appear to pass through this point smoothly, as was seen by Merrill et al. (2006) and not ending in a singularity, as seen by Merkin (1985). We indicate this disruption point at $\lambda = 0$ in the numerical solutions by $\circ$ in the plots of $f''(0)$, Figs. 2a, 3a, 4a, 5a. This non-existence of a lower branch solution at $\lambda = 0$ is more clearly seen in the plots of $\theta'(0)$, Figs. 2b, 3b, 4b, 5b, where $|\theta'(0)|$ appears to become infinite as $|\lambda| \to 0$.

We now consider some limiting forms for large values of a given parameter, starting with the case when there is strong fluid withdrawal through the surface.

4 Some limiting forms

4.1 Strong suction

To obtain a solution to (13, 14) valid for $s \gg 1$, we put

$$f = s + s^{-1} F, \quad \zeta = s\,\eta \tag{20}$$

Equations 13, 14 then become

$$F''' + (1 + s^{-2} F)F'' + s^{-2}(F' - F'^2) + s^{-2}\,|A|\left(F' - 1 + \frac{\zeta}{2} F''\right) = 0 \tag{21}$$

$$F(0) = 0, \quad \Lambda\, F''(0) = s^{-1}\left[F'(0) - (1 + \lambda)\right], \quad F' \to 1 \text{ as } \zeta \to \infty \tag{22}$$

where primes now denote differentiation with respect to ζ.

When Λ, and the other parameters, are of $O(1)$, we look for a solution by expanding

$$F(\zeta; s) = \zeta + s^{-1} F_1 + \cdots \tag{23}$$

giving

$$F_1''' + F_1'' = 0, \quad F_1(0) = 0, \quad \Lambda F_1''(0) = -\lambda, \quad F_1' \to 0 \text{ as } \zeta \to \infty \tag{24}$$

The required solution is

$$F_1 = \frac{\lambda}{\Lambda}(1 - e^{-\zeta}), \quad \text{so that } F_1''(0) = -\frac{\lambda}{\Lambda} \quad \text{and} \quad \theta = \frac{e^{-\zeta}}{\Lambda} s^{-1} \tag{25}$$

so that

$$\left(\frac{d^2 f}{d\eta^2}\right)_{\eta=0} \sim -\frac{\lambda}{\Lambda} + \cdots, \quad \left(\frac{d\theta}{d\eta}\right)_{\eta=0} \sim -\frac{1}{\Lambda} + \cdots, \quad \left(\frac{d f}{d\eta}\right)_{\eta=0} \sim 1 + \frac{\lambda}{\Lambda} s^{-1} + \cdots \quad \text{as } s \to \infty \tag{26}$$

Expression (25) shows that the effect of strong suction is to give an $O(s^{-1})$ perturbation to the free stream and to reduce the overall temperature to $O(s^{-1})$. When $\Lambda > 0$, the skin friction approaches a constant (negative) value for large s.

Expression (26) also shows that, if Λ is sufficiently small, we can get reversed flow when $\lambda < 0$ in this limit. To examine this case further, we now assume that Λ is small, of $O(s^{-1})$, by putting $\Lambda = \alpha\, s^{-1}$, with α of $O(1)$. The leading-order problem is again Eq. 24 but now subject to

$$\alpha\, F_1''(0) = F_1'(0) - (1 + \lambda), \quad F_1' \to 1 \text{ as } \zeta \to \infty \tag{27}$$

This gives

$$F_1' = 1 + \frac{\lambda}{1 + \alpha} e^{-\zeta}, \quad F_1 = \zeta + \frac{\lambda}{1 + \alpha}(1 - e^{-\zeta}) \tag{28}$$

Expression (28) gives $F_1'(0) = 1 + \lambda/(1+\alpha)$, thus giving reversed flow in the opposing flow case if $\lambda < -(1+\alpha)$ or $\lambda < -(1+\Lambda\, s)$ when Λ is small.

4.2 Free convection limit

There are two cases to consider in the limit of large λ, namely $\Lambda > 0$ and $\Lambda = 0$. For the first case, we apply the transformation

$$f = \lambda^{1/3}\, G, \quad \xi = \lambda^{1/3}\, \eta \tag{29}$$

This results in

$$G''' + G\,G'' - G'^2 + \lambda^{-2/3}\, G' + |A|\lambda^{-2/3}\left(G' + \frac{\xi}{2}\, G''\right) - |A|\lambda^{-4/3} = 0 \tag{30}$$

now subject to

$$G(0) = \lambda^{-1/3}\, s, \quad \Lambda\, G''(0) = -1 + \lambda^{-1/3}\, G'(0) - \lambda^{-1}, \quad G' \to \lambda^{-2/3} \text{ as } \xi \to \infty \tag{31}$$

where primes now denote differentiation with respect to ξ. The leading-order problem in an expansion in $\lambda^{-1/3}$ has the solution, on the assumption that both s and $|A|$ are of $O(1)$,

$$G(\xi) = \Lambda^{-1/3}\,(1 - \mathrm{e}^{-\xi/\Lambda^{1/3}}) \tag{32}$$

hence

$$f''(0) \sim -\frac{\lambda}{\Lambda} + \cdots, \quad \theta'(0) \to -\Lambda^{-1} \quad \text{as } \lambda \to \infty \tag{33}$$

Expression (33) indicates that there will be a problem as $\Lambda \to 0$ and to deal with the $\Lambda = 0$ case, we put

$$f = \lambda^{1/2}\, \overline{G}, \quad \overline{\xi} = \lambda^{1/2}\, \eta \tag{34}$$

The leading-order problem is the same as that given in (30) though now subject to $\overline{G}'(0) = 1$. The solution is $\overline{G} = (1 - \mathrm{e}^{-\overline{\xi}})$, giving

$$f''(0) \sim -\lambda^{3/2} + \cdots, \quad \theta'(0) \sim -\lambda^{1/2} + \cdots \quad \text{as } \lambda \to \infty \tag{35}$$

The difference in behaviour of $f''(0)$ and $\theta'(0)$ for large λ depending on whether $\Lambda \neq 0$, with expression (33) giving $f''(0)$ decreasing linearly with λ and $\theta'(0)$ approaching a constant value, or $\Lambda = 0$, where expression (35) gives $f''(0)$ of $O(\lambda^{3/2})$ and $\theta'(0)$ large of $O(\lambda^{1/2})$ can be seen in Fig. 3.

4.3 Large temperature slip

To obtain a solution valid for large Λ we put, on assuming that $s = 0$,

$$f = \eta + \Lambda^{-1}\, H \tag{36}$$

and leave η unscaled. This then gives

$$H''' + \left(1 + \frac{|A|}{2}\right)\eta\, H'' - (1 - |A|)H' + \Lambda^{-1}\,(H H'' - H'^2) = 0 \tag{37}$$

where primes again denote differentiation with respect to η. The leading-order problem for H' is in effect to forced convection problem (18) though now subject to

$$H''(0) = -\lambda, \quad H' \to 0 \text{ as } \eta \to \infty \tag{38}$$

The solution to (37, 38) can be expressed in terms of confluent hypergeometric functions Slater (1960) as

$$H' = \frac{\lambda\Gamma\left(\frac{4-|A|}{2(2+|A|)}\right)}{(2+|A|)^{1/2}\sqrt{\pi}}\,\mathrm{e}^{-(2+|A|)\eta^2/4}\; U\left(\frac{4-|A|}{2(2+|A|)};\ \frac{1}{2};\ \frac{(2+|A|)\eta^2}{4}\right) \tag{39}$$

We note that, for steady flow, $A = 0$, expression (39) gives

$$H' = \frac{\lambda}{\sqrt{2\pi}}\left(\mathrm{e}^{-\eta^2/2} - \eta\int_{\eta}^{\infty}\mathrm{e}^{-t^2/2}\,\mathrm{d}t\right) \tag{40}$$

From (36, 38) $f''(0) \sim -\dfrac{\lambda}{\Lambda} + \cdots$ for large Λ and, for steady flow, expression (40) shows that $f'(0) \sim 1 + \dfrac{\lambda}{\sqrt{2\pi}}\,\Lambda^{-1} + \cdots$. Both these expressions show that the main effect of having a large temperature slip is for the flow to be dominated by the outer flow with buoyancy and the unsteadiness giving only a perturbation to this state.

4.4 Solution for large $|A|$

This case, from (12), corresponds to the highly time-dependent situation and to obtain a solution of (13, 14) valid for large $|A|$, we put

$$f = \eta + |A|^{-1}\,h, \quad Y = |A|^{1/2}\,\eta \tag{41}$$

Equation 13 becomes

$$h''' + \frac{Y}{2}\,h'' + h' + |A|^{-1}\,(Yh'' - h') + |A|^{-3/2}(h\,h'' - h'^2) = 0 \tag{42}$$

now subject to the boundary conditions, again on restricting attention to the case with $s = 0$,

$$h(0) = 0, \quad \Lambda h''(0) = -\lambda + |A|^{-1/2}\,h'(0), \quad h' \to 0 \ \text{ as } \ Y \to \infty \tag{43}$$

Here primes denote differentiation with respect to Y. The solution to the leading-order problem for h_0 in an expansion in inverse powers of $|A|$ is

$$h_0' = -\frac{\lambda}{\Lambda}\,Y\mathrm{e}^{-Y^2/4}, \quad h_0 = -\frac{2\lambda}{\Lambda}\,(1 - \mathrm{e}^{-Y^2/4}) \tag{44}$$

From (41, 44), we then have

$$f''(0) \sim -\frac{\lambda}{\Lambda} + \cdots, \quad \theta'(0) \sim -\frac{1}{\Lambda}\cdots \quad \text{as } \ |A| \to \infty \tag{45}$$

The transformation for η in (41) and the form for $\theta'(0)$ in (45) are consistent with (18, 19) in the limit of $|A| \to \infty$.

Transformation (41) gives $Y = y/(\alpha_{\mathrm{m}}t)^{1/2}$ for large $|\gamma|$, being the usual independent variable for time-dependent growth through diffusion. This, together with the form for f in (41), indicates that the effect of unsteadiness is for the boundary layer to grow through diffusion from the outer flow already established at $t = 0$. Expressions (45) show that the effect of having a temperature slip is for both $f''(0)$ and $\theta'(0)$ to approach constant values as $|A|$ increases.

5 Conclusions

We have studied the problem of the unsteady mixed convection boundary-layer flow near a two-dimensional stagnation point on a vertical permeable surface embedded in a porous medium with suction and with temperature slip effects. Using an appropriate similarity transformation, the governing boundary-layer equations were transformed into a set of nonlinear ordinary differential equations which we then solved numerically using a shooting method with the aid of shootlib function from Maple software. We show that the present results are in a good agreement with previously published results in the literature. The results for the skin friction $f''(0)$ and the Nusselt number $-\theta'(0)$ are graphically presented and the effects of the temperature slip parameter Λ, the suction parameter s and the unsteadiness parameter A on both $f''(0)$ and $\theta'(0)$ are discussed for both assisting and opposing flow regimes. Dual solutions were found to exist in assisting flow. For opposing flow, dual solutions exist only up to a critical point with no solution existing beyond this point. It was found that increasing the temperature slip parameter Λ and the suction parameter s increased the range of existence of a solution, whereas increasing the unsteadiness parameter A decreased the range of solution existence.

We also considered some limiting asymptotic forms for strong suction, $s \gg 1$, the free convection limit, $\lambda \to \infty$ and large temperature slip, $\Lambda \gg 1$ and unsteadiness, $|A| \gg 1$. For strong suction, we found that there was a difference in the nature of the solution depending on whether Λ was of $O(1)$ or small, see expressions (26) and (28). We found the possibility of reversed flow when Λ was small. The nature of the free convection limit also depended on whether $\Lambda > 0$ or $\Lambda = 0$, see transformations (29) and (34) with both $f''(0)$ and $\theta'(0)$ having different asymptotic forms in each case, noting that $f''(0)$ and $\theta'(0)$ approach constant limits as $\lambda \to \infty$ if $\Lambda > 0$, whereas they continue to grow as λ is increased if $\Lambda = 0$. For both strong temperature slip and unsteadiness, the solution is a perturbation to the basic outer flow. For large $|A|$, the boundary layer grows through diffusion at a rate proportional to $(\alpha_\mathrm{m}\, t)^{1/2}$ from the initial state.

Acknowledgements AMR and SA gratefully acknowledge the financial supports received in the form of research grants: Research University Grant (1001/PMATHS/811166) from Universiti Sains Malaysia and FRGS from the Ministry of Higher Education, Malaysia. The authors also wish to thank the referees for their helpful comments and suggestions.

References

Ali, M.E., Magyari, E.: Unsteady fluid and heat flow induced by a submerged stretching surface while its steady motion is slowed down gradually. Int. J. Heat Mass Transf. **50**, 188–195 (2007)

Aly, E.H., Elliott, L., Ingham, D.B.: Mixed convection boundary-layer flow over a vertical surface embedded in a porous medium. Eur. J. Mech. B **22**, 529–543 (2003)

Andersson, H.I., Aarseth, J.B., Dandapat, B.S.: Heat transfer in a liquid film on an unsteady stretching surface. Int. J. Heat Mass Transf. **43**, 69–74 (2000)

Birkhoff, G.: Hydrodynamics, a Study in Fact and Similitude, Revised edn. Princeton University Press, Princeton (1960)

Cheng, P.: Similarity solutions for mixed convection from horizontal impermeable surfaces in saturated porous media. Int. J. Heat Mass Transf. **20**, 893–898 (1977)

Fang, T.-G., Zhang, J., Yao, S.-S.: Viscous flow over an unsteady shrinking sheet with mass transfer. Chin. Phys. Lett. **26**, 014703-1–014703-4 (2009)

Harris, S.D., Ingham, D.B., Pop, I.: Transient free convection on a vertical plate subjected to a change in surface heat flux in porous media. Fluid Dynamics Res. **18**, 313–324 (1996)

Harris, S.D., Ingham, D.B., Pop, I.: Thermal capacity effect on transient free convection adjacent to a vertical surface in a porous medium. Transp. Porous Media **46**, 1–18 (2002)

Harris, S.D., Ingham, D.B., Pop, I.: Mixed convection boundary-layer flow near the stagnation point on a vertical surface in a porous medium: Brinkman model with slip. Transp. Porous Media **77**, 267–285 (2009)

Ingham, D.B., Pop, I. (eds.): Transport Phenomena in Porous Media III. Elsevier, Oxford (2005)

Merkin, J.H.: Mixed convection boundary layer flow on a vertical surface in a saturated porous medium. J. Eng. Math. **14**, 301–313 (1980)

Merkin, J.H.: On dual solutions occurring in mixed convection in a porous medium. J. Eng. Math. **20**, 171–179 (1985)

Merrill, K., Beauchesne, M., Previte, J., Paullet, J., Weidman, P.: Final steady flow near a stagnation point on a vertical surface in a porous medium. Int. J. Heat Mass Transf. **49**, 4681–4686 (2006)

Mukhopadhyay, S., Andersson, H.: Effects of slip and heat transfer analysis of flow over an unsteady stretching surface. Heat Mass Transf. **45**, 1447–1452 (2009)

Nazar, R., Amin, N., Pop, I.: Unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium. Int. J. Heat Mass Transfer **47**, 2681–2688 (2004)

Nield, D.A., Bejan, A.: Convection in Porous Media, 3rd edn. Springer, New York (2006)

Pop, I., Ingham, D.B.: Convective Heat Transfer: Mathematical and Computational Modeling of Viscous Fluids and Porous Media. Pergamon Press, Oxford (2001)

Slater, L.J.: Confluent hypergeometric functions. Cambridge University Press, Cambridge (1960)

Tie-Gang, F., Ji, Z., Shan-Shan, Y.: Viscous flow over an unsteady shrinking sheet with mass transfer. Chin. Phys. Let. **26**, 014703-1–014703-4 (2009)

Vadasz, P. (ed.): Emerging Topics in Heat and Mass Transfer in Porous Media. Springer, Berlin (2008)

Vafai, K. (ed.): Handbook of Porous Media, 2nd edn. Taylor & Francis, New York (2005)

Vafai, K.: Porous Media: Applications in Biological Systems and Biotechnology. CRC Press, Tokyo (2010)

Wang, C.Y.: Liquid film on an unsteady stretching surface. Q. Appl. Math. **48**, 601–610 (1990)

Yang, K.T.: Unsteady laminar boundary layers in an incompressible stagnation flow, Trans. ASME J. Appl. Mech. **25**, 421–427 (1958)