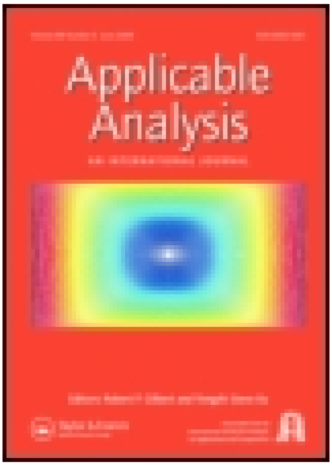

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Group classification and exact solutions of generalized modified Boussinesq equation

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We present symmetry classification and exact solutions of generalized modified Boussinesq (GMB) equation. The direct method of group classification is utilized to determine four different functional forms of $f(u)$. The GMB equation admits two-dimensional principle algebra for arbitrary $f(u)$ and the algebra extends to three-dimensional for other forms of $f(u)$. Similarity reductions are made in each case and exact solutions are derived.

Keywords: generalized modified Boussinesq equation; group classification; similarity solutions

AMS Subject Classifications: 70G65; 35B06

1. Introduction

Joseph Valentin Boussinesq (1842–1929) derived an equation for the propagation of long waves on the surface of water with a small amplitude. There have been several generalizations of the Boussinesq equation such as the improved Boussinesq equation, modified Boussinesq equation, or the dispersive water wave. Here we study the generalized modified Boussinesq (GMB) equation which describes the nonlinear model of longitudinal wave propagation of elastic rods and is governed by [1,2]

$$u_{tt} - \delta u_{ttxx} - (f(u))_{xx} = 0, \quad (f_{uu} \neq 0) \tag{1}$$

where δ is a nonzero constant and $f(u)$ is an arbitrary function. It plays an important role in nonlinear lattice waves, iron sound waves and vibrations in a nonlinear string. The arbitrary functions arise in differential equations can be obtained from physical laws or experiments but in some cases they cannot be deduced. The method of group classification can be used to determine the forms of functions in those cases. The GMB equation has been studied extensively using numerical and analytical approaches. Bogolubsky [3] derived exact solitary wave solutions of GMB equation for $f(u) = b_1u + b_2u^{p+1} + b_3u^{2p+1}$

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when $p = 2, 3, 5$. The explicit solitary wave solutions of GMB equation for $f(u) = b_1u + b_2u^3 + b_3u^5$ and $f(u) = b_1u + b_2u^2 + b_3u^3$ using the method of solving algebraic equations were derived by Li and Zhang [4], Zhang and Ma [5]. In [1], the homotopy perturbation method is used to derive the approximate solutions of GMB equation. This approach involve approximations and solution may not converge. A similar problem arises in numerical schemes as well. The Lie classical and nonclassical methods are used to derive some exact solutions of different classes of Boussinesq equations, for good account of these see e.g. [6–8] and references therein.

In this manuscript, we give symmetry classification, optimal system, and exact solutions of GMB equation. The symmetry group method is one of the most efficient instrument for solving linear and nonlinear partial differential equations. We utilize the group classification method to determine the four functional form of $f(u)$. We show that when $f(u)$ is arbitrary then GMB equation possesses two-dimensional algebra. For other forms of $f(u)$, the Lie algebra of GMB equation is extended to three-dimensional. We construct the optimal system [9,10] for each form of $f(u)$ and also derive the independent exact solutions.

2. Symmetry classification and exact solutions of generalized modified Boussinesq equation

In this section, we provide complete classification for classical Lie symmetries and exact solutions of Equation (1). The vector field of the Lie point symmetries is

$$X = \xi^1(t,\ x,\ u)\frac{\partial}{\partial t} + \xi^2(t,\ x,\ u)\frac{\partial}{\partial x} + \phi(t,\ x,\ u)\frac{\partial}{\partial u}. \tag{2}$$

The Lie point symmetry generators of Equation (1) are found by applying the invariance condition [10–12]

$$X^{[4]}[u_{tt} - \delta u_{ttxx} - (f(u))_{xx}]\ |_{(1)} = 0, \tag{3}$$

where

$$\begin{aligned} X^{[4]} = X &+ \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{ttt}\frac{\partial}{\partial u_{ttt}} \\ &+ \zeta_{ttx}\frac{\partial}{\partial u_{ttx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{tttt}\frac{\partial}{\partial u_{tttt}} + \zeta_{tttx}\frac{\partial}{\partial u_{tttx}} \\ &+ \zeta_{ttxx}\frac{\partial}{\partial u_{ttxx}} + \zeta_{txxx}\frac{\partial}{\partial u_{txxx}} + \zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}}. \end{aligned} \tag{4}$$

In Equation (4) ζs are given by

$$\begin{aligned} \zeta_i &= D_i(\phi) - u_j D_i(\xi^j), \\ \zeta_{ij} &= D_j(\zeta_i) - u_{il} D_j(\xi^l), \\ \zeta_{ijk} &= D_k(\zeta_{ij}) - u_{ij\,l} D_k(\xi^l), \\ \zeta_{ijkm} &= D_m(\zeta_{ijk}) - u_{ijkl} D_m(\xi^l), \quad i, j, k, l = t, x, \end{aligned} \tag{5}$$

where D_m is the total derivative operator. Equation (3) after expansion and then separation with respect to the powers of different derivatives of u yields following over determined system in unknown coefficients ξ^1, ξ^2 and ϕ:

$$\xi^1_u = 0, \quad \xi^1_x = 0, \quad \xi^2_u = 0, \quad \xi^2_t = 0, \tag{6}$$
$$\xi^2_{xx} = 0, \quad \xi^1_{tt} - 2\phi_{tu} = 0, \tag{7}$$
$$\phi_{uu} = 0, \quad \phi_{tt} = 0, \quad \phi_x = 0, \tag{8}$$
$$2(\delta - 1)\xi^1_t + (1 - \delta)\phi_u + 2\delta\xi^2_x = 0, \tag{9}$$
$$2(1 - \delta)\xi^2_x f_u + (\delta - 1)\phi_u f_u - \phi f_{uu} - 2\delta\xi^1_t f_u = 0, \tag{10}$$
$$2(1 - \delta)\xi^2_x f_{uu} + (\delta - 2)\phi_u f_{uu} - \phi f_{uuu} - 2\delta\xi^1_t f_{uu} = 0. \tag{11}$$

If $f(u)$ is arbitrary in u then Equation (1) admits

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x},$$

that forms two-dimensional principle algebra. Now we investigate all the possibilities of $f(u)$ for which extension of the principle algebra is possible.

Substituting $2\delta\xi^1_t + 2\delta\xi^2_x = (\delta - 1)\phi_u + 2\xi^1_t$ from Equation (9) in Equations (10) and (11), we obtain

$$2f_u(\xi^2_x - \xi^1_t) - \phi f_{uu} = 0, \tag{12}$$
$$2f_{uu}(\xi^2_x - \xi^1_t) - \phi f_{uuu} - \phi_u f_{uu} = 0. \tag{13}$$

After simplification of Equations (12) and (13), we have

$$\phi f^2_{uu} - \phi f_u f_{uuu} + \phi_u f_u f_{uu} = 0. \tag{14}$$

Differentiating Equation (14) and subsequent elimination of ϕ gives

$$f_u f_{uu} f_{uuuu} + f^2_{uu} f_{uuu} - 2 f_u f^2_{uuu} = 0. \tag{15}$$

Replacing $g = f_u$ in Equation (15) transforms to

$$\frac{g_{uuu}}{g_{uu}} + \frac{g_u}{g} - 2\frac{g_{uu}}{g_u} = 0. \tag{16}$$

Solving Equation (16) for g and then using relation $g = f_u$ lead to the following four forms of $f(u)$ (see [13])

$$(a) \quad f(u) = \alpha e^{\beta u} + \gamma, \tag{17}$$
$$(b) \quad f(u) = \alpha u^2 + \beta u + \gamma, \tag{18}$$
$$(c) \quad f(u) = \frac{\ln(\alpha u + \beta)}{\alpha} + \gamma, \tag{19}$$
$$(d) \quad f(u) = (\alpha u + \beta)^n + \gamma, \quad n \neq 0, 1, 2, \tag{20}$$

where α, β and γ are arbitrary constants.
Now we find the symmetry algebras for each form of $f(u)$ in the following cases.

Case I: Lie Symmetries, optimal systems and exact solutions of (1) for $f(u) = \alpha e^{\beta u} + \gamma$

If $f(u) = \alpha e^{\beta u} + \gamma$, then Equation (1) becomes

$$u_{tt} - \delta u_{ttxx} - \alpha\beta^2 e^{\beta u} u^2_x - \alpha\beta e^{\beta u} u_{xx} = 0. \tag{21}$$

Table 1. Commutator table of the Lie algebra of Equation (21).

$[Y_i, Y_j]$	Y_1	Y_2	Y_3
Y_1	0	0	βX_1
Y_2	0	0	0
Y_3	$-\beta X_1$	0	0

Table 2. Adjoint representation of Lie algebra of Equation (21).

Ad	Y_1	Y_2	Y_3
X_1	X_1	X_2	$X_3 - \beta\gamma X_1$
X_2	X_1	X_2	X_3
X_3	$e^{\beta\gamma} X_1$	X_2	X_3

Solving the determining Equations (6)–(11) with $f(u) = \alpha e^{\beta u} + \gamma$ lead to the following three Lie point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \beta t \frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}. \tag{22}$$

In this case, the principle algebra is extended to three-dimensional. Now we find the optimal systems of these operators.[9,10] The commutation relation between these operators is given in Table 1.

The adjoint representation of the Lie algebra of Equation (21) is defined as

$$Ad(exp(\gamma X_i)X_j) = X_j - \gamma[X_i, X_j] + \frac{1}{2}\gamma^2[X_i, [X_i, X_j]] - .... \tag{23}$$

Using Equation (23) we find the all adjoint representation of Lie algebra of Equation (21) listed in Table 2.

The optimal system of one-dimensional subalgebra admitted by Equation (21) are [9,10]

$$X_1, \quad X_3, \quad cX_1 + X_2, \quad X_2 + cX_3, \quad c \neq 0. \tag{24}$$

Next we use this optimal system to find the exact solutions of Equation (21).
Solution of (21) using X_1: The characteristic equation corresponding to X_1 is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}, \tag{25}$$

which turns into

$$r = x, \quad s = t, \quad v(r) = u(t, x). \tag{26}$$

Equation (21) with the use of (26) reduces to

$$\beta v_r^2 + v_{rr} = 0. \tag{27}$$

The solution of Equation (27) is

$$v(r) = \frac{\ln(\beta c r + \beta d)}{\beta}. \tag{28}$$

Thus

$$u(t,x) = \frac{\ln(\beta c x + \beta d)}{\beta} \tag{29}$$

is a solution of Equation (21) invariant under X_1.
Solution of (21) using X_3: Using $X_3 = \beta t \frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}$ we obtain

$$r = x, \quad s = \frac{\ln(t)}{\beta}, \quad v(r) = \frac{2\ln(t) + \beta u(t,x)}{\beta}. \tag{30}$$

Equation (21) in terms of new coordinates gives rise to

$$2 - \alpha\beta^3 e^{\beta v} v_r^2 - \alpha\beta^2 e^{\beta v} v_{rr} = 0, \tag{31}$$

which can be integrated to obtain exact solution of Equation (21) as

$$u(t,x) = \frac{1}{\beta}\ln\left(\frac{x^2 - \alpha\beta^2 c x + \alpha\beta^2 d}{\alpha\beta t^2}\right). \tag{32}$$

Solution of (21) using $X = cX_1 + X_2$: The similarity variables for operator $X = c\partial/\partial t + \partial/\partial x$ are

$$r = \frac{cx - t}{c}, \quad s = \frac{t}{c}, \quad v(r) = u(t,x), \tag{33}$$

which gives the following version of Equation (21)

$$v_{rr} - \delta v_{rrrr} - \alpha\beta^2 c^2 e^{\beta v} v_r^2 - \alpha\beta c^2 e^{\beta v} v_{rr} = 0. \tag{34}$$

This implies

$$v(r) - \delta v_{rr} - \alpha c^2 e^{\beta v} = 0 \tag{35}$$

and the solution is given by the following integral equation

$$\pm\int \frac{\beta\delta}{\sqrt{\delta\beta^2 v^2 - 2\alpha\beta\delta c^2 e^{\beta v} + \beta^2\delta^2 c_1}} dv - r - c_2 = 0, \tag{36}$$

where $r = \frac{cx-t}{c}$ and $v(r) = u(t,x)$.
Solution of (21) using $X = X_2 + cX_3$: The similarity variables for operator $X = \partial/\partial x + c(\beta t \frac{\partial}{\partial t} - 2\frac{\partial}{\partial u})$ are

$$r = \frac{-\ln(t) + c\beta x}{c\beta}, \quad s = \frac{\ln(t)}{c\beta}, \quad v(r) = \frac{2\ln(t) + \beta u(t,x)}{\beta}, \tag{37}$$

which gives the following version of Equation (21)

$$(1 - c\beta\delta)v_{rr} - \delta v_{rrrr} - \alpha\beta^4 c^2 e^{\beta v} v_r^2 - \alpha\beta^3 c^2 e^{\beta v} v_r + c\beta v_r = 0. \tag{38}$$

This implies

$$(1 - c\beta\delta)v_r - \delta v_{rrr} - \alpha\beta^3 c^2 e^{\beta v} v_r + c\beta v = 0. \tag{39}$$

The Lie symmetries, optimal system, reduced form, and exact solutions for other cases for $f(u)$ of modified Boussinesq equation are presented in Table 3.

Table 3. Lie symmetries, optimal system, reduced form, and exact solutions for different form of $f(u)$ of modified Boussinesq equation.

Modified Boussinesq equation:	$u_{tt} - \delta u_{ttxx} - (f(u))_{xx} = 0$
Case II	
$f(u)$	$\alpha u^2 + \beta u + \gamma$
GMB equation	$u_{tt} - \delta u_{ttxx} - 2\alpha u_x^2 - (2\alpha u + \beta)u_{xx} = 0$
Lie point symmetries	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = -\alpha t \frac{\partial}{\partial t} + (2\alpha u + \beta)\frac{\partial}{\partial u}$
Optimal system	$X_1, \quad X_3, \quad cX_1 + X_2, \quad X_2 + cX_3, \quad c \neq 0$
Reduction via X_1	$2\alpha v_r^2 + (2\alpha v + \beta)v_{rr} = 0$
Solution via X_1	$u(t,x) = \frac{\sqrt{\beta^2 + 4\alpha c x + 4\alpha d} - \beta}{2\alpha}$
Reduction via X_3	$6v - (6\delta + 2\alpha v)v_{rr} - 2\alpha v_r^2 = 0$
Solution via X_3	$u(t,x) = \frac{\alpha^2 c^2 + 3\delta + 2\alpha c x + x^2 - \beta t^2}{2\alpha t^2}$
Reduction via $cX_1 + X_2$	$(1 - \beta c^2)v - \delta v_{rr} - \alpha c^2 v^2 = 0$
Solution via $cX_1 + X_2$	$\pm \int \frac{\sqrt{3}\delta}{\sqrt{3\delta v^2 - 3\beta\delta c^2 v^2 - 2\alpha\delta c^2 v^3 + 3\delta^2 c_1}} dv - r - c_2 = 0,$ $r = \frac{cx - t}{c}, \quad v(r) = u(t,x)$
Reduction via $X_2 + cX_3$	$(1 - 2\delta\alpha^2 c^2 - 2\alpha^3 c^2 v)v_{rr} - \delta v_{rrrr} + 3\delta\alpha c v_{rrr}$ $-3\alpha c v_r - 2\alpha^3 c^2 v_r^2 + 2\alpha^2 c^2 v = 0$
Case III	
$f(u)$	$\frac{\ln(\alpha u + \beta)}{\alpha} + \gamma$
GMB equation	$(\alpha u + \beta)^2 u_{tt} - \delta(\alpha u + \beta)^2 u_{ttxx} + \alpha u_x^2 - (\alpha u + \beta)u_{xx} = 0$
Lie point symmetries	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \alpha t \frac{\partial}{\partial t} + (2\alpha u + 2\beta)\frac{\partial}{\partial u}$
Optimal system	$X_1, \quad X_3, \quad cX_1 + X_2, \quad X_2 + cX_3, \quad c \neq 0$
Reduction via X_1	$\alpha v_r^2 - (\alpha v(r) + \beta)v_{rr} = 0$
Solution via X_1	$u(t,x) = \frac{e^{\alpha d x + \beta e} - \beta}{\alpha}$
Reduction via X_3	$2\alpha^2 v^3 - (2\alpha^2\delta v^2 + \alpha v)v_{rr} + \alpha v_r^2 = 0$
Solution via X_3	$\frac{\mp\frac{1}{2}\alpha\delta \ln\left[\frac{2\alpha^2 + 4\alpha^3\delta v}{\sqrt{4\alpha^3\delta}} + \sqrt{4\alpha^3\delta v^2 + 4\alpha^2 v + \frac{1}{c_1^2}}\right]\sqrt{4}}{\sqrt{\alpha^3\delta}}$

(*Continued*)

Table 3. (*Continued*).

Modified Boussinesq equation:	$u_{tt} - \delta u_{ttxx} - (f(u))_{xx} = 0$
	$\pm c_1 \ln\left[\frac{\frac{2}{c_1^2} + 4\alpha^2 v + \frac{2\sqrt{4\alpha^3\delta v^2 + 4\alpha^2 v + \frac{1}{c_1^2}}}{c_1}}{v}\right] -$ $r - c_2 = 0,\ r = x,\ v(r) = \frac{\alpha u(t,x)+\beta}{\alpha t^2}$
Reduction via $cX_1 + X_2$	$v - \delta v_{rr} - \frac{c^2 \ln(\alpha v+\beta)}{\alpha} = 0$
Solution via $cX_1 + X_2$	$\pm \int \frac{\alpha\delta}{\sqrt{\alpha^2\delta v^2 - 2\alpha\delta c^2 v(\ln(\alpha v+\beta)) - 2\beta\delta c^2(\ln(\alpha v+\beta)) + 2c^2\delta\alpha v + 2c^2\beta\delta + c_1\alpha^2\delta^2}} dv$ $-r - c_2 = 0,\ \ r = \frac{cx-t}{c}, v(r) = u(t,x)$
Reduction via $X_2 + cX_3$	$2\alpha^2c^2v^3 - 3\alpha c v^2 v_r + \alpha c^2 v_r^2 + (v^2 + 2\alpha^2c^2v^2 - \alpha c^2 v)v_{rr}$ $-3\alpha c v^2 v_{rrr} - \delta v^2 v_{rrrr} = 0$
Case IV	
$f(u)$	$(\alpha u + \beta)^n + \gamma$
GMB equation	$u_{tt} - \delta u_{ttxx} - \alpha^2 n(n-1)(\alpha u+\beta)^{n-2}u_x^2$ $-\alpha n(\alpha u + \beta)^{n-1}u_{xx} = 0$
Lie point symmetries	$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = -\alpha(n-1)t\frac{\partial}{\partial t} + (2\alpha u + 2\beta)\frac{\partial}{\partial u}$
Optimal system	$X_1,\quad X_3,\quad cX_1 + X_2,\quad X_2 + cX_3,\quad c \neq 0$
Reduction via X_1	$\alpha(n-1)v_r^2 + (\alpha v + \beta)v_{rr} = 0$
Solution via X_1	$u(t,x) = \frac{(\alpha dnx + \alpha en)^{\frac{1}{n}} - \beta}{\alpha}$
Reduction via X_3	$2(n+1)v - (2\delta(n+1) + n\alpha^n(n-1)^2v^{n-1})v_{rr}$ $-n\alpha^n(n-1)^3v^{n-2}v_r^2 = 0$
Solution via $X_3,\ n = 3$	$u(t,x) = \frac{3d - \sqrt{3}x - 3\sqrt{\alpha}\beta t}{3\alpha^{\frac{3}{2}}t}$
Reduction via $cX_1 + X_2$	$v - \delta v_{rr} - c^2(\alpha v + \beta)^n = 0$
Solution via $cX_1 + X_2$	$\pm \int \frac{\alpha\delta(n+1)}{\sqrt{(\alpha\delta(n+1))(\alpha n v^2 + \alpha v^2 - 2c^2(\alpha v+\beta)^{n-1} + \alpha\delta c_1 n + \alpha\delta c}} dv$ $-r - c_2 = 0\quad r = \frac{cx-t}{c}, v(r) = u(t,x)$
Reduction via $X_2 + cX_3$	$(1 + 2\alpha^2 - nc^2\alpha^{n+2}v^{n-1})v_{rr} - (2\alpha^2c + \alpha c(n+1))v_r$ $-(2\alpha^2c + \alpha c(n+1))v_{rrr} - \delta v_{rrrr} + c^2\alpha^n v^{n-2}v_r^2 + 2\alpha^2 cv = 0$

3. Conclusions

In this article, we discussed group classification and exact solutions GMB equation by using Lie theory. We classified four forms of the function involved in GMB equation. We construct Lie point symmetries with respect to different forms of $f(u)$. These forms have exponential, quadratic, logarithmic and power law form. We found four exact solutions with exponential form, three exact solutions with quadratic form, three exact solutions with logarithmic form, and three exact solution with power law form of $f(u)$. Most solutions are of explicit form and one integral form in each case. To the best of our knowledge, the solutions obtained here are not obtained in literature. The derived solutions cannot be interpreted physically due to lack of experimental sources, however these solutions will play an essential role for numerical simulations in applied mathematics.

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