My Question : please variation this action with respect to tensor metric $g_{\mu\nu}$.this called the Einstein equation. To obtain the Einstein equation, we vary the action with respect to the metric tensor

$$s = \int d^4x \sqrt{-g}[R + \alpha(\varphi)R_{GB}]$$

$$R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

where R is the Ricci scalar, R_{μν} is a Ricci tensor, R_{μνρσ} is a Riemann tensor,

g is the determinant of the metric tensor,α(φ) is a function of φ(r) ,R_{GB} is a quadratic Gauss-bonnet (GB) term, G is the gravitational constant, and (d^4x) represents the volume element in four-dimensional spacetime.

α(φ) = f(φ)

φ=φ(r)

Let us now turn our attention to the third term δS_3 in (19.34), in which we must express $\delta\sqrt{-g}$ in terms of the variation $\delta g^{\mu\nu}$. Recalling that $g = \det[g_{\mu\nu}]$, we note that the cofactor of the element $g_{\mu\nu}$ in this determinant is $gg_{\mu\nu}$. It follows that

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu},$$

where in the second equality we have used the result (19.33). Thus, we have

$$\delta\sqrt{-g} = -\tfrac{1}{2}(-g)^{-1/2}\delta g = -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

I see different results in papers, but I don't have any code in Maple for them. I need the Maple code for this equation to continue deriving my equation.

so i give you some example of this result in next page :

Example 1: (arXiv:1812.06941v1 [hep-th] 17 Dec 2018)

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi + f(\phi)R^2_{GB} - 2\Lambda\right].$$

$$R^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,,$$

by varying the action (S) with respect to the metric tensor $g_{\mu\nu}$, we derive the gravitationalfiel equations:

$$G_{\mu\nu} = T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor, with the latter having the form

$$T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(g_{\rho\mu}g_{\lambda\nu} + g_{\lambda\mu}g_{\rho\nu}\right)\eta^{\kappa\lambda\alpha\beta}\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\nabla_\gamma\partial_\kappa f(\phi) - \Lambda\, g_{\mu\nu}\,.$$

example2 : (arXiv:hep-th/0504052v2 25 May 2005)

The starting action is given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)G\right\}\ .$$

On the other hand, the variation over the metric $g_{\mu\nu}$ gives

$$\begin{aligned} 0 \ =\ & \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \gamma\left(\frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right) + \frac{1}{2}g^{\mu\nu}\left(-V(\phi) + f(\phi)G\right) \\ & -2f(\phi)RR^{\mu\nu} + 2\nabla^\mu\nabla^\nu\left(f(\phi)R\right) - 2g^{\mu\nu}\nabla^2\left(f(\phi)R\right) \\ & +8f(\phi)R^\mu{}_\rho R^{\nu\rho} - 4\nabla_\rho\nabla^\mu\left(f(\phi)R^{\nu\rho}\right) - 4\nabla_\rho\nabla^\nu\left(f(\phi)R^{\mu\rho}\right) \\ & +4\nabla^2\left(f(\phi)R^{\mu\nu}\right) + 4g^{\mu\nu}\nabla_\rho\nabla_\sigma\left(f(\phi)R^{\rho\sigma}\right) - 2f(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4\nabla_\rho\nabla_\sigma\left(f(\phi)R^{\mu\rho\sigma\nu}\right)\ . \end{aligned}$$